

# Observed galaxy number counts on the lightcone up to second order: III. Magnification Bias

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We study up to second order the galaxy number over-density that depends on magnification in redshift space on cosmological scales for a concordance model. The result contains all general relativistic effects up to second order which arise from observing on the past light cone, including all redshift and lensing distortions, contributions from velocities, Sachs-Wolfe, integrated SW and time-delay terms. We find several new terms and contributions that could be potentially important for an accurate calculation of the bias on estimates of non-Gaussianity and on precision parameter estimates.

## I. INTRODUCTION

Galaxy catalogues in observational cosmology depend on the apparent flux from the source. Practically, this can be translated into the dependence of the observed number counts on the magnification that modifies the spatial distribution of the sources (for example of galaxies or quasars) (see, for example, [1–16]).

At the same time, on cosmological scales, relativistic effects alter the observed number over-density giving the well-known corrections both from the usual redshift space distortions and gravitational lensing convergence [4], Doppler, Sachs-Wolfe, integrated SW and time-delay type terms. The full relativistic effects have been studied at first order in perturbation theory by [17–21] and at second order by [24–27]. (A similar analysis at second order has been done on the cosmological distance-redshift relation in [28]–[36], or to the weak lensing in [37–39].)

This paper completes our studies on derivation of the formula for the observed galaxy number over-density up to second order on cosmological scales. Following the “cosmic rulers” approach of [21, 40] at first order and of [25] at second order, we generalize the previous results obtained in [24, 25] adding the magnification bias dependence (see also [26]).

In the last part of the work, we look at the galaxy number over-density up to second order in redshift space on cosmological scales for a concordance model, in the Poisson gauge. The result contains all general relativistic effects up to second order that arise from observing on the past light cone, including all redshift effects, lensing distortions from convergence and shear, contributions from velocities, Sachs-Wolfe, integrated SW and time-delay terms. We find several new terms and contributions which could be important for the accurate calculation on estimates of the primordial non-Gaussianity [41].

The paper is organised in the following way: in Section II we briefly review and generalize the cosmic rulers to obtain the second-order perturbations of galaxy number counts in redshift space; in Section II we analyze the magnification both at first and second order, and, in Section III we present the galaxy number over-density up to second order with all relativistic effects that arise from observing on the past lightcone and as a function of the magnification bias. We perturb a flat Robertson-Walker universe in the Poisson Gauge for a  $\Lambda$ CDM model in Section IV. Finally, Section V is devoted to conclusions.

Conventions: units  $c = G = 1$ ; signature is  $(-, +, +, +)$ ; Greek indices run over 0, 1, 2, 3, and Latin over 1, 2, 3.

## II. COSMIC LABORATORY

In this section we summarize briefly the cosmic rulers defined in Ref. [25] (at first order see also [21, 40]). First of all, it is useful to define the parallel and perpendicular projection operators to the observed line-of-sight direction (see also [21, 40]). For any spatial vectors and tensors:

$$A_{\parallel} = n^i n^j A_{ij} \ , \quad B_{\perp}^i = \mathcal{P}^{ij} B_j = B^i - n^i B_{\parallel} \ , \quad (1)$$

where  $\mathcal{P}_j^i = \delta_j^i - n^i n_j$ . The directional derivatives are defined as

$$\partial_{\parallel} = n^i \frac{\partial}{\partial \bar{x}^i} \ , \quad \partial_{\parallel}^2 = \partial_{\parallel} \partial_{\parallel} \ , \quad \partial_{\perp i} = \mathcal{P}_i^j \partial_j = \frac{\partial}{\partial \bar{x}^i} - n_i \partial_{\parallel} \ , \ , \quad \frac{\partial n^j}{\partial \bar{x}^i} = \frac{1}{\bar{\chi}} \mathcal{P}_i^j \ , \quad (2)$$

and we have

$$\begin{aligned} \frac{\partial B^i}{\partial \bar{x}^j} &= n^i n_j \partial_{\parallel} B_{\parallel} + n^i \partial_{\perp j} B_{\parallel} + \partial_{\perp j} B_{\perp}^i + n_j \partial_{\parallel} B_{\perp}^i + \frac{1}{\bar{\chi}} \mathcal{P}_j^i B_{\parallel} , \\ \text{and} \quad \nabla_{\perp}^2 &= \partial_{\perp i} \partial_{\perp}^i = \delta^{ij} \frac{\partial}{\partial \bar{x}^i} \frac{\partial}{\partial \bar{x}^j} - \partial_{\parallel}^2 - \frac{2}{\bar{\chi}} \partial_{\parallel} . \end{aligned} \quad (3)$$

Redshift-space or *redshift frame* is the “cosmic laboratory” where we probe the observations. In redshift-space we use coordinates which effectively flatten our past lightcone so that the photon geodesic from an observed galaxy has the following conformal space-time coordinates [21, 25, 40]:

$$\bar{x}^{\mu} = (\bar{\eta}, \bar{\mathbf{x}}) = (\eta_0 - \bar{\chi}, \bar{\chi} \mathbf{n}). \quad (4)$$

Here  $\bar{\chi}(z)$  is the comoving distance to the observed redshift in redshift-space,  $\mathbf{n}$  is the observed direction to the galaxy, i.e.  $n^i = \bar{x}^i / \bar{\chi} = \delta^{ij} (\partial \bar{\chi} / \partial \bar{x}^j)$ . Using  $\bar{\chi}$  as an affine parameter in the redshift frame, the total derivative along the past light cone is  $d/d\bar{\chi} = -\partial/\partial \bar{\eta} + n^i \partial/\partial \bar{x}^i$ . Defining the photon 4-momentum  $p^{\mu} = \nu(a)k^{\mu}/a$ , where  $a$  is the scale factor,  $\nu \propto 1/a$  is the frequency, we perturb the comoving null geodesic vector  $k^{\mu}$ , at second order, in the following way

$$k^{\mu}(\bar{\chi}) = \frac{dx^{\mu}}{d\bar{\chi}}(\bar{\chi}) = \frac{d}{d\bar{\chi}}(\bar{x}^{\mu} + \delta x^{\mu})(\bar{\chi}) = \left( -1 + \delta\nu^{(1)} + \frac{1}{2}\delta\nu^{(2)}, n^i + \delta n^{i(1)} + \frac{1}{2}\delta n^{i(2)} \right)(\bar{\chi}), \quad (5)$$

where in the redshift frame at zero order

$$\bar{k}^{\mu} = \frac{d\bar{x}^{\mu}}{d\bar{\chi}} = (-1, \mathbf{n}). \quad (6)$$

Defining  $x^{\mu}(\chi)$  as the coordinates in the *physical frame*, where  $\chi$  is the physical comoving distance of the source, we can set up a mapping between redshift space and real space (the “physical frame”) in the following way

$$\chi = \bar{\chi} + \delta\chi \quad \text{where} \quad \delta\chi = \delta\chi^{(1)} + \frac{1}{2}\delta\chi^{(2)}, \quad (7)$$

$$x^{\mu}(\chi) = \bar{x}^{\mu}(\bar{\chi}) + \Delta x^{\mu}(\bar{\chi}) \quad \text{where} \quad \Delta x^{\mu}(\bar{\chi}) = \Delta x^{\mu(1)}(\bar{\chi}) + \frac{1}{2}\Delta x^{\mu(2)}(\bar{\chi}). \quad (8)$$

Taking into account the observed redshift  $(1+z)|_{\chi} = (u_{\mu}p^{\mu})|_{\chi}/(u_{\mu}p^{\mu})|_o$ , and assuming that  $a_o = 1$  we can obtain explicitly the scale factor [25]

$$\Delta \ln a^{(1)} = -\delta\nu^{(1)} - E_{\hat{0}\hat{0}}^{(1)} + E_{\hat{0}\parallel}^{(1)}, \quad (9)$$

$$\begin{aligned} \Delta \ln a^{(2)} &= -\delta\nu^{(2)} - E_{\hat{0}\hat{0}}^{(2)} + E_{\hat{0}\parallel}^{(2)} + 2E_{\hat{0}\parallel}^{(1)} \left( \delta\nu^{(1)} + \delta n_{\parallel}^{(1)} \right) + 2 \left( \delta x^{0(1)} + \delta x_{\parallel}^{(1)} \right) \partial_{\parallel} \left( E_{\hat{0}\parallel}^{(1)} - E_{\hat{0}\hat{0}}^{(1)} \right) \\ &\quad - \frac{2}{\mathcal{H}} \left( E_{\hat{0}\parallel}^{(1)} - E_{\hat{0}\hat{0}}^{(1)} \right) \left( \frac{d \Delta \ln a}{d\bar{\chi}} \right)^{(1)} + 2 \left[ - \left( E_{\hat{0}\parallel}^{(1)} - E_{\hat{0}\hat{0}}^{(1)} \right) + \frac{1}{\mathcal{H}} \left( \frac{d \Delta \ln a}{d\bar{\chi}} \right)^{(1)} \right] \delta\nu^{(1)} \\ &\quad - 2\delta x^{0(1)} \left( \frac{d\delta\nu}{d\bar{\chi}} \right)^{(1)} + 2E_{\hat{0}\perp i}^{(1)} \delta n_{\perp}^{i(1)} + 2 \left[ \partial_{\perp i} \left( E_{\hat{0}\parallel}^{(1)} - E_{\hat{0}\hat{0}}^{(1)} \right) - \frac{1}{\bar{\chi}} E_{\hat{0}\perp i}^{(1)} \right] \delta x_{\perp}^{i(1)}, \end{aligned} \quad (10)$$

and the comoving distance

$$\delta\chi^{(1)} = \delta x^{0(1)} - \Delta x^{0(1)}, \quad (11)$$

$$\delta\chi^{(2)} = \delta x^{0(2)} - \frac{1}{\mathcal{H}} \Delta \ln a^{(2)} + \frac{(\mathcal{H}' + \mathcal{H}^2)}{\mathcal{H}^3} \left( \Delta \ln a^{(1)} \right)^2 - \frac{2}{\mathcal{H}} \delta\nu^{(1)} \Delta \ln a^{(1)} + 2\delta\nu^{(1)} \delta x^{0(1)}, \quad (12)$$

where we have defined the parallel and perpendicular parts of the tetrad in the comoving frame<sup>1</sup>:

$$E_{\hat{\alpha}}^i = n^i E_{\hat{\alpha}}^{\parallel} + E_{\hat{\alpha}}^{\perp i}, \quad \text{where} \quad E_{\hat{\alpha}}^{\parallel} = n_i E_{\hat{\alpha}}^i \quad \text{and} \quad E_{\hat{\alpha}}^{\perp i} = \mathcal{P}_j^i E_{\hat{\alpha}}^j, \quad (13)$$

$$E_{\hat{\alpha}i} = n_i E_{\hat{\alpha}\parallel} + E_{\hat{\alpha}\perp i}, \quad \text{where} \quad E_{\hat{\alpha}\parallel} = n^i E_{\hat{\alpha}i} \quad \text{and} \quad E_{\hat{\alpha}\perp i} = \mathcal{P}_i^j E_{\hat{\alpha}j}. \quad (14)$$

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<sup>1</sup> Note that in general  $E_{\hat{\alpha}}^i$  is not a 3-space tensor in the index  $i$ , so that  $E_{\hat{0}}^{\parallel} \neq E_{\hat{0}\parallel}$  and  $E_{\hat{0}}^{\perp i} \neq \delta^{ij} E_{\hat{0}\perp j}$ .

Here, up to second order, the scale factor is

$$a(x^0(\chi)) = \bar{a} \left( 1 + \Delta \ln a^{(1)} + \frac{1}{2} \Delta \ln a^{(2)} \right), \quad (15)$$

where  $\bar{a} = a(\bar{x}^0(\bar{\chi})) = 1/(1+z)$ , prime is  $\partial/\partial \bar{x}^0 = \partial/\partial \eta$  and  $\mathcal{H} = \bar{a}'/\bar{a}$ . Then, we have, at first order,

$$\Delta x^{0(1)} = \frac{\Delta \ln a^{(1)}}{\mathcal{H}}, \quad (16)$$

$$\Delta x_{\parallel}^{(1)} = \delta x^{0(1)} + \delta x_{\parallel}^{(1)} - \Delta x^{0(1)}, \quad (17)$$

$$\Delta x_{\perp}^{i(1)} = \delta x_{\perp}^{i(1)}, \quad (18)$$

and, at second order,

$$\Delta x^{0(2)} = \frac{1}{\mathcal{H}} \Delta \ln a^{(2)} - \frac{(\mathcal{H}' + \mathcal{H}^2)}{\mathcal{H}^3} \left( \Delta \ln a^{(1)} \right)^2, \quad (19)$$

$$\Delta x_{\parallel}^{(2)} = \delta x^{0(2)} + \delta x_{\parallel}^{(2)} - \Delta x^{0(2)} + 2 \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) \left( \delta x^{0(1)} - \frac{\Delta \ln a^{(1)}}{\mathcal{H}} \right), \quad (20)$$

$$\Delta x_{\perp}^{i(2)} = 2 \delta n_{\perp}^{i(1)} \delta x^{0(1)} - \frac{2}{\mathcal{H}} \delta n_{\perp}^{i(1)} \Delta \ln a^{(1)} + \delta x_{\perp}^{i(2)}. \quad (21)$$

Here let us also define  $\Delta x_{\perp i}^{(n)} = \delta_{ij} \Delta x_{\perp}^{j(n)}$ .

The next task is to study the physical number density of galaxies  $n_g$  as a function of the physical comoving coordinates  $x^\mu$  and the magnification  $\mathcal{M}$ . In particular, we consider the cumulative physical number density sample with a flux larger than a observed limit  $\bar{\mathcal{F}}$  which can be translated in terms of a the inferred threshold luminosity  $\bar{L}(z)$  ( $n_g = N$  in [20, 23]). The physical number density contained within a volume  $\bar{\mathcal{V}}$  is given by

$$\begin{aligned} \mathcal{N} = & \int_{\bar{\mathcal{V}}} \sqrt{-g(x^\alpha)} n_g(x^\alpha, \mathcal{M}) \varepsilon_{\mu\nu\rho\sigma} u^\mu(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3} d^3 \bar{\mathbf{x}} = \\ & \int_{\bar{\mathcal{V}}} \sqrt{-\hat{g}(x^\alpha)} a^3(x^0) n_g(x^\alpha, \mathcal{M}) \varepsilon_{\mu\nu\rho\sigma} E_0^\mu(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3} d^3 \bar{\mathbf{x}}, \end{aligned} \quad (22)$$

where  $\varepsilon_{\mu\nu\rho\sigma}$  is the Levi-Civita tensor,  $\sqrt{-\hat{g}} = \sqrt{-g}/a^4$ ,  $\hat{g}^{\mu\nu}$  the comoving metric and  $u^\mu = E_0^\mu/a$  is the four velocity vector as a function of comoving location. In the redshift frame is, by definition,

$$\mathcal{N} = \int_{\bar{\mathcal{V}}} \bar{a}^3(\bar{x}^0) n_g(\bar{x}^0, \bar{\mathbf{x}}, \bar{L}) d^3 \bar{\mathbf{x}}. \quad (23)$$

Using the cosmic rulers we get

$$\sqrt{-\hat{g}(x^\alpha)} = 1 + \Delta \sqrt{-\hat{g}(\bar{x}^\alpha)}^{(1)} + \frac{1}{2} \Delta \sqrt{-\hat{g}(\bar{x}^\alpha)}^{(2)}, \quad (24)$$

where<sup>2</sup>

$$\Delta \sqrt{-\hat{g}(\bar{x}^\alpha)}^{(1)} = \frac{1}{2} \hat{g}_\mu^{\mu(1)}(\bar{x}^\alpha), \quad (25)$$

$$\Delta \sqrt{-\hat{g}(\bar{x}^\alpha)}^{(2)} = \frac{1}{4} \hat{g}_\mu^{\mu(1)}(\bar{x}^\alpha) \hat{g}_\nu^{\nu(1)}(\bar{x}^\alpha) + \frac{1}{2} \hat{g}_\mu^{\mu(2)}(\bar{x}^\alpha) - \frac{1}{2} \hat{g}_\mu^{\nu(1)}(\bar{x}^\alpha) \hat{g}_\nu^{\mu(1)}(\bar{x}^\alpha) + \left( \frac{\partial \hat{g}_\mu^\mu}{\partial \bar{x}^\nu} \right)^{(1)}(\bar{x}^\alpha) \Delta x^{\nu(1)}, \quad (26)$$

From Eq. (15),

$$a^3 = \bar{a}^3 \left[ 1 + 3 \Delta \ln a^{(1)} + 3 \left( \Delta \ln a^{(1)} \right)^2 + \frac{3}{2} \Delta \ln a^{(2)} \right]. \quad (27)$$

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<sup>2</sup> Here we define  $\hat{g}_\mu^{\mu(n)} = \hat{g}^{\mu\nu(0)} \hat{g}_{\mu\nu}^{(n)}$ .

Defining

$$V(x^\alpha) = \varepsilon_{\mu\nu\rho\sigma} E_{\hat{0}}^\mu(x^\alpha) \frac{\partial x^\nu}{\partial \bar{x}^1} \frac{\partial x^\rho}{\partial \bar{x}^2} \frac{\partial x^\sigma}{\partial \bar{x}^3} = 1 + \Delta V(\bar{x}^\alpha)^{(1)} + \frac{1}{2} \Delta V(\bar{x}^\alpha)^{(2)}, \quad (28)$$

we find, at first and second order,

$$\Delta V(\bar{x}^\alpha)^{(1)} = E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + \partial_{\parallel} \Delta x_{\parallel}^{(1)} + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2\kappa^{(1)}, \quad (29)$$

$$\begin{aligned} \Delta V(\bar{x}^\alpha)^{(2)} &= E_{\hat{0}}^{0(2)} + E_{\hat{0}}^{\parallel(2)} + \partial_{\parallel} \Delta x_{\parallel}^{(2)} + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(2)} - 2\kappa^{(2)} + 2 \left( \kappa^{(1)} \right)^2 - 2|\gamma^{(1)}|^2 + \vartheta_{ij}^{(1)} \vartheta^{ij(1)} + \frac{2}{\bar{\chi}^2} \left( \Delta x_{\parallel}^{(1)} \right)^2 \\ &+ \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \partial_{\parallel} \Delta x_{\parallel}^{(1)} - \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \kappa^{(1)} - 4\kappa^{(1)} \partial_{\parallel} \Delta x_{\parallel}^{(1)} + \frac{2}{\bar{\chi}} \Delta x_{\perp i}^{(1)} \partial_{\parallel} \Delta x_{\perp}^{i(1)} - 2 \left( \partial_{\parallel} \Delta x_{\perp}^{i(1)} \right) \left( \partial_{\perp i} \Delta x_{\parallel}^{(1)} \right) \\ &+ \frac{2}{\bar{\mathcal{H}}} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right)' \Delta \ln a^{(1)} + 2\partial_{\parallel} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\parallel}^{(1)} + 2\partial_{\perp i} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\perp}^{i(1)} \\ &+ 2E_{\hat{0}}^{0(1)} \partial_{\parallel} \Delta x_{\parallel}^{(1)} - 2E_{\hat{0}}^{\parallel(1)} \partial_{\parallel} \Delta x_{\parallel}^{(1)} - 4 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \kappa^{(1)} - 2E_{\hat{0}}^{\perp i(1)} \partial_{\perp i} \left( \Delta x_{\parallel}^{(1)} + \Delta x_{\perp}^{i(1)} \right) \\ &+ \frac{4}{\bar{\chi}} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\parallel}^{(1)}, \end{aligned} \quad (30)$$

where

$$\kappa^{(n)} = -\frac{1}{2} \partial_{\perp i} \Delta x_{\perp}^{i(n)} \quad (31)$$

is the coordinate weak lensing convergence term at order  $n$ ,

$$\gamma_{ij}^{(1)} = -\partial_{\perp(i} \Delta x_{\perp j)}^{(1)} - \mathcal{P}_{ij} \kappa^{(1)} \quad (32)$$

is the coordinate weak lensing shear term<sup>3</sup>,  $\gamma_{ij}^{(1)} \gamma^{ij(1)} = 2|\gamma^{(1)}|^2$  and  $\vartheta_{ij}^{(1)} = -\partial_{\perp[i} \Delta x_{\perp j]}^{(1)}$ . From Eq. (8), we note that

$$\partial_{\parallel} \Delta x^{\mu(n)}(\bar{\chi}, \mathbf{n}) = \partial_{\bar{\chi}} \Delta x^{\mu(n)}, \quad (33)$$

where  $\partial_{\bar{\chi}}$  is applied to all terms that are functions of  $\bar{x}^0 = \bar{\eta}(\bar{\chi})$  and/ or  $\bar{x}^i = \bar{x}^i(\bar{\chi})$ .

The next subsection will be devoted to obtain at second order the magnification and Sec. III we will analyze in detail  $n_g$  in order to obtain the observed galaxy over-density.

### A. Magnification

The magnification is defined in the following way

$$\mathcal{M} = \left( \frac{\mathcal{D}_A}{\bar{\mathcal{D}}_A} \right)^{-2} = \left( \frac{\mathcal{D}_L}{\bar{\mathcal{D}}_L} \right)^{-2}, \quad (34)$$

where  $\mathcal{D}_L$  is the luminosity distance,  $\mathcal{D}_A$  is the angular distance which are related by

$$\mathcal{D}_L = (1+z)^2 \mathcal{D}_A, \quad \text{and} \quad \bar{\mathcal{D}}_A = \bar{a}(\bar{x}^0(\bar{\chi})) \bar{\chi}. \quad (35)$$

Here  $\bar{\mathcal{D}}_L$  and  $\bar{\mathcal{D}}_A$  are luminosity and angular distance at zero order.

Using the cosmic rulers prescription, let us define  $(\mathcal{D}_A/\bar{\mathcal{D}}_A)^2$  in the following way [21]

$$\mathcal{M}^{-1} = \left( \frac{\mathcal{D}_A}{\bar{\mathcal{D}}_A} \right)^2 = \frac{\sqrt{-g(x^\alpha)}}{\bar{a}(\bar{x}^0(\bar{\chi}))^2} \varepsilon_{\mu\nu\rho\sigma} u^\mu v^\nu \frac{\partial x^\rho}{\partial \bar{x}^i} \frac{\partial x^\sigma}{\partial \bar{x}^j} \alpha^i \beta^j, \quad (36)$$

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<sup>3</sup> Here  $A_{[i} B_{j]} = (A_i B_j - B_i A_j)/2$  and  $A_{(i} B_{j)} = (A_i B_j + B_i A_j)/2$ .

where  $\{n^i, \alpha^j, \beta^k\}$  is a three dimensional orthonormal basis<sup>4</sup> and  $v^\nu$  is the unit spatial part of the null vector  $p^\mu$ , orthogonal to  $u^\mu$  and directed away from the observer, i.e.

$$v^\mu = \frac{p^\mu}{p^\rho u_\rho} + u^\mu = \frac{k^\mu}{k^\rho u_\rho} + u^\mu, \quad (37)$$

where we have used  $p^\mu = \nu k^\mu / a$ . Now, making the change of the variable from  $\chi$  to  $\bar{\chi}$ , we obtain

$$k^\mu(\chi) = \frac{dx^\mu(\chi)}{d\chi} = \left( \frac{d\bar{\chi}}{d\chi} \right) \frac{dx^\mu(\chi)}{d\bar{\chi}} \quad (38)$$

and

$$v^\mu = \frac{(dx^\mu(\chi)/d\bar{\chi})}{(dx^\rho(\chi)/d\bar{\chi}) u_\rho} + u^\mu. \quad (39)$$

Here  $dx^\rho(\chi)/d\bar{\chi}$  is equivalent to  $\partial_{\bar{\chi}} x^\rho(\chi(\bar{\chi}), \mathbf{n})$  [21, 25]. Taking in to account that  $u^\mu = E_{\hat{0}}^\mu / a$ ,  $u_\mu = a E_{\hat{0}\mu}$  and, using Eq.(39), Eq. (36) turns out

$$\begin{aligned} \mathcal{M}^{-1} &= \frac{\sqrt{-g(x^\alpha)}}{(dx^\rho(\chi)/d\bar{\chi} u_\rho) \bar{a}(\bar{x}^0(\bar{\chi}))^2} \varepsilon_{\mu\nu\rho\sigma} u^\mu \frac{dx^\nu(\chi)}{d\bar{\chi}} \frac{\partial x^\rho}{\partial \bar{x}^j} \frac{\partial x^\sigma}{\partial \bar{x}^k} \alpha^j \beta^k \\ &= \frac{\sqrt{-\hat{g}(x^\alpha)}}{(dx^\rho(\chi)/d\bar{\chi} E_{\hat{0}\rho})} \left( \frac{a(x^0)}{\bar{a}(\bar{x}^0)} \right)^2 \varepsilon_{\mu\nu\rho\sigma} E_{\hat{0}}^\mu \frac{\partial x^\nu}{\partial \bar{x}^i} \frac{\partial x^\rho}{\partial \bar{x}^j} \frac{\partial x^\sigma}{\partial \bar{x}^k} n^i \alpha^j \beta^k. \end{aligned} \quad (40)$$

We note immediately that Eq. (40) generalizes for any gauges the relation obtained in [21]. Using the cosmic rulers, we will expand the relation to second order this relation in order to obtain  $\mathcal{M}$  and  $\mathcal{D}_A$  (or  $\mathcal{D}_L$ ). Let us point out that, at second order,  $\mathcal{D}_L$  is also already obtained with different methods in [28]–[35] and [26].

Using the relations obtained in Sec. II, it easy to proof that, up to second order,

$$\varepsilon_{\mu\nu\rho\sigma} E_{\hat{0}}^\mu \frac{\partial x^\nu}{\partial \bar{x}^i} \frac{\partial x^\rho}{\partial \bar{x}^j} \frac{\partial x^\sigma}{\partial \bar{x}^k} n^i \alpha^j \beta^k = 1 + \Delta V(\bar{x}^\alpha)^{(1)} + \frac{1}{2} \Delta V(\bar{x}^\alpha)^{(2)} \quad (41)$$

and

$$\left( \frac{dx^\rho(\chi)}{d\bar{\chi}} E_{\hat{0}\rho} \right) = 1 + \Delta \left( \frac{dx^\rho(\chi)}{d\bar{\chi}} E_{\hat{0}\rho} \right)^{(1)} + \frac{1}{2} \Delta \left( \frac{dx^\rho(\chi)}{d\bar{\chi}} E_{\hat{0}\rho} \right)^{(2)}, \quad (42)$$

where

$$\Delta \left( \frac{dx^\rho(\chi)}{d\bar{\chi}} E_{\hat{0}\rho} \right)^{(1)} = -\partial_{\bar{\chi}} \Delta x^{0(1)} - E_{\hat{0}0}^{(1)} + E_{\hat{0}\parallel}^{(1)}, \quad (43)$$

$$\begin{aligned} \Delta \left( \frac{dx^\rho(\chi)}{d\bar{\chi}} E_{\hat{0}\rho} \right)^{(2)} &= -\partial_{\bar{\chi}} \Delta x^{0(2)} - E_{\hat{0}0}^{(2)} + E_{\hat{0}\parallel}^{(2)} - \frac{2}{\mathcal{H}} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right)' \Delta \ln a^{(1)} - 2\partial_{\parallel} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\parallel}^{(1)} \\ &\quad - 2\partial_{\perp i} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\perp}^{i(1)} + 2E_{\hat{0}0}^{(1)} \partial_{\bar{\chi}} \Delta x^{0(1)} + 2E_{\hat{0}\parallel}^{(1)} \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} + 2E_{\hat{0}\perp i}^{(1)} \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} \\ &\quad - \frac{2}{\bar{\chi}} E_{\hat{0}\perp i}^{(1)} \Delta x_{\perp}^{i(1)}. \end{aligned} \quad (44)$$

Then, Eq. (40) yields

$$\mathcal{M}^{-1} = 1 + \Delta (\mathcal{M}^{-1})^{(1)} + \frac{1}{2} \Delta (\mathcal{M}^{-1})^{(2)}, \quad (45)$$

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<sup>4</sup> e.g.  $n^i = \epsilon^{ijk} \alpha_j \beta_k$ ,  $\alpha^i \alpha_i = \beta^i \beta_i = 1$  etc.

where

$$\begin{aligned}
\Delta (\mathcal{M}^{-1})^{(1)} &= \frac{1}{2} \hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + 2\Delta \ln a^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2\kappa^{(1)}, \\
\Delta (\mathcal{M}^{-1})^{(2)} &= \frac{1}{2} \hat{g}_\mu^{\mu(2)} + E_{\hat{0}}^{0(2)} + E_{\hat{0}}^{\parallel(2)} + E_{\hat{0}0}^{(2)} - E_{\hat{0}\parallel}^{(2)} + 2\Delta \ln a^{(2)} + \partial_{\bar{\chi}} \left( \Delta x^{0(2)} + \Delta x_{\parallel}^{(2)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(2)} - 2\kappa^{(2)}, \\
&\quad + \frac{1}{4} \hat{g}_\mu^{\mu(1)} \hat{g}_\nu^{\nu(1)} - \frac{1}{2} \hat{g}_\mu^{\nu(1)} \hat{g}_\nu^{\mu(1)} + 2 \left( \kappa^{(1)} \right)^2 - 4\kappa^{(1)} \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} - \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \kappa^{(1)} - 4 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \kappa^{(1)} \\
&\quad + 2 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \right) \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2 \left( E_{\hat{0}0}^{(1)} + E_{\hat{0}}^{\parallel(1)} \right) \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) \\
&\quad + \frac{2}{\bar{\chi}^2} \left( \Delta x_{\parallel}^{(1)} \right)^2 + 2 \left[ E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2\kappa^{(1)} \right] \\
&\quad \times \left( \frac{1}{2} \hat{g}_\mu^{\mu(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \Delta x^{0(1)} \right) + 2\partial_{\parallel} \left( \frac{1}{2} \hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\parallel}^{(1)} \\
&\quad - 2|\gamma^{(1)}|^2 + \vartheta_{ij}^{(1)} \vartheta^{ij(1)} + 2 \left( \frac{1}{\bar{\chi}} \Delta x_{\perp i}^{(1)} - \partial_{\perp i} \Delta x_{\parallel}^{(1)} \right) \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} + \frac{4}{\bar{\chi}} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\parallel}^{(1)} \\
&\quad + 2\partial_{\perp i} \left( \frac{1}{2} \hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\perp}^{i(1)} + 2E_{\hat{0}\perp i}^{(1)} \left( \frac{1}{\bar{\chi}} \Delta x_{\perp}^{i(1)} - \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} \right) \\
&\quad - 2E_{\hat{0}}^{\perp i(1)} \partial_{\perp i} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\mathcal{H}} \left( \frac{1}{2} \hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right)' \Delta \ln a^{(1)} \\
&\quad + 4 \left[ \frac{1}{2} \hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2\kappa^{(1)} \right] \Delta \ln a^{(1)} \\
&\quad + 2 \left( \Delta \ln a^{(1)} \right)^2.
\end{aligned} \tag{46}$$

Consequently the magnification turns out

$$\mathcal{M} = 1 + \Delta \mathcal{M}^{(1)} + \frac{1}{2} \Delta \mathcal{M}^{(2)}, \tag{48}$$

where

$$\begin{aligned}
\Delta \mathcal{M}^{(1)} &= -\Delta (\mathcal{M}^{-1})^{(1)} = -\frac{1}{2}\hat{g}_\mu^{\mu(1)} - E_{\hat{0}}^{0(1)} - E_{\hat{0}}^{\parallel(1)} - E_{\hat{0}\hat{0}}^{(1)} + E_{\hat{0}\parallel}^{(1)} - 2\Delta \ln a^{(1)} - \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) - \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \\
&\quad + 2\kappa^{(1)}, \tag{49} \\
\Delta \mathcal{M}^{(2)} &= -\Delta (\mathcal{M}^{-1})^{(2)} + 2 \left[ \Delta (\mathcal{M}^{-1})^{(1)} \right]^2 = -\frac{1}{2}\hat{g}_\mu^{\mu(2)} - E_{\hat{0}}^{0(2)} - E_{\hat{0}}^{\parallel(2)} - E_{\hat{0}\hat{0}}^{(2)} + E_{\hat{0}\parallel}^{(2)} - 2\Delta \ln a^{(2)} - \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(2)} \\
&\quad - \partial_{\bar{\chi}} \left( \Delta x^{0(2)} + \Delta x_{\parallel}^{(2)} \right) + 2\kappa^{(2)} + \frac{1}{4} \left( \hat{g}_\mu^{\mu(1)} \right)^2 + \frac{1}{2} \hat{g}_\mu^{\nu(1)} \hat{g}_\nu^{\mu(1)} + \frac{6}{\bar{\chi}^2} \left( \Delta x_{\parallel}^{(1)} \right)^2 + 2|\gamma^{(1)}|^2 - \vartheta_{ij}^{(1)} \vartheta^{ij(1)} \\
&\quad + 2 \left[ E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \right] \left[ E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) \right] \\
&\quad + 2\partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} + \hat{g}_\mu^{\mu(1)} \left[ E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \right] \\
&\quad + 2 \left( E_{\hat{0}\hat{0}}^{(1)} + E_{\hat{0}}^{\parallel(1)} \right) \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) - 2\partial_{\parallel} \left( \frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\parallel}^{(1)} \\
&\quad - 4\kappa^{(1)} \left[ \frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) \right] - \frac{12}{\bar{\chi}} \kappa^{(1)} \Delta x_{\parallel}^{(1)} + 6 \left( \kappa^{(1)} \right)^2 \\
&\quad - 2 \left( \frac{1}{\bar{\chi}} \Delta x_{\perp i}^{(1)} - \partial_{\perp i} \Delta x_{\parallel}^{(1)} \right) \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} - 2\partial_{\perp i} \left( \frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\perp}^{i(1)} \\
&\quad - 2E_{\hat{0}\perp i}^{(1)} \left( \frac{1}{\bar{\chi}} \Delta x_{\perp}^{i(1)} - \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} \right) + 2E_{\hat{0}}^{\perp i(1)} \partial_{\perp i} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) \\
&\quad - \frac{2}{\mathcal{H}} \left( \frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right)' \Delta \ln a^{(1)} + 6 \left( \Delta \ln a^{(1)} \right)^2 \\
&\quad + 4 \left[ \frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2\kappa^{(1)} \right] \Delta \ln a^{(1)}. \tag{50}
\end{aligned}$$

In similar way, we can obtain the luminosity distance

$$\frac{\mathcal{D}_L}{\mathcal{D}_L} = 1 + \frac{\mathcal{D}_L^{(1)}}{\mathcal{D}_L} + \frac{1}{2} \frac{\mathcal{D}_L^{(2)}}{\mathcal{D}_L}, \tag{51}$$

where

$$\begin{aligned}
\frac{\mathcal{D}_L^{(1)}}{\mathcal{D}_L} &= \frac{\mathcal{D}_A^{(1)}}{\mathcal{D}_A} = \frac{1}{2} \Delta (\mathcal{M}^{-1})^{(1)} = -\frac{1}{2} \Delta \mathcal{M}^{(1)} = \frac{1}{4} \hat{g}_\mu^{\mu(1)} + \frac{1}{2} E_{\hat{0}}^{0(1)} + \frac{1}{2} E_{\hat{0}}^{\parallel(1)} + \frac{1}{2} E_{\hat{0}\hat{0}}^{(1)} - \frac{1}{2} E_{\hat{0}\parallel}^{(1)} + \Delta \ln a^{(1)} \\
&\quad + \frac{1}{2} \partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + \frac{1}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - \kappa^{(1)} \tag{52}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathcal{D}_L^{(2)}}{\mathcal{D}_L} = \frac{\mathcal{D}_A^{(2)}}{\mathcal{D}_A} &= \frac{1}{2}\Delta(\mathcal{M}^{-1})^{(2)} - \frac{1}{4}\left[\Delta(\mathcal{M}^{-1})^{(1)}\right]^2 = -\frac{1}{2}\Delta\mathcal{M}^{(2)} + \frac{3}{4}\left(\Delta\mathcal{M}^{(1)}\right)^2 = \frac{1}{4}\hat{g}_\mu^{(2)} + \frac{1}{2}E_0^{(2)} + \frac{1}{2}E_0^{\parallel(2)} \\
&+ \frac{1}{2}E_{00}^{(2)} - \frac{1}{2}E_{0\parallel}^{(2)} + \Delta\ln a^{(2)} + \frac{1}{2}\partial_{\bar{\chi}}\left(\Delta x^{0(2)} + \Delta x_{\parallel}^{(2)}\right) + \frac{1}{\bar{\chi}}\Delta x_{\parallel}^{(2)} - \kappa^{(2)} + \frac{1}{16}\hat{g}_\mu^{\mu(1)}\hat{g}_\nu^{\nu(1)} \\
&- \frac{1}{4}\hat{g}_\mu^{\nu(1)}\hat{g}_\nu^{\mu(1)} - \kappa^{(1)}\left[E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)} + \partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right)\right] - \frac{1}{2}\hat{g}_\mu^{\mu(1)}\kappa^{(1)} \\
&+ \frac{1}{4}\hat{g}_\mu^{\mu(1)}\left[E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)} + \partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right) + \frac{2}{\bar{\chi}}\Delta x_{\parallel}^{(1)}\right] \\
&- \partial_{\bar{\chi}}\Delta x_{\parallel}^{(1)}\partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right) - \left(E_{00}^{(1)} + E_{\hat{0}}^{\parallel(1)}\right)\partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right) - |\gamma^{(1)}|^2 + \frac{1}{2}\vartheta_{ij}^{(1)}\vartheta^{ij(1)} \\
&+ \frac{1}{\bar{\chi}}\Delta x_{\parallel}^{(1)}\left[E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)} + \partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right)\right] + \frac{1}{4}\left[-\left(E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)}\right)\right. \\
&+ 3\left(E_{00}^{(1)} - E_{0\parallel}^{(1)}\right) + 3\partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right)\left.\right]\left[E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)} + \partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right)\right] \\
&+ \partial_{\parallel}\left(\frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)}\right)\Delta x_{\parallel}^{(1)} + \left(\frac{1}{\bar{\chi}}\Delta x_{\perp i}^{(1)} - \partial_{\perp i}\Delta x_{\parallel}^{(1)}\right)\partial_{\bar{\chi}}\Delta x_{\perp}^{i(1)} \\
&+ \partial_{\perp i}\left(\frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)}\right)\Delta x_{\perp}^{i(1)} + E_{0\perp i}^{(1)}\left(\frac{1}{\bar{\chi}}\Delta x_{\perp}^{i(1)} - \partial_{\bar{\chi}}\Delta x_{\perp}^{i(1)}\right) \\
&- E_{\hat{0}}^{\perp i(1)}\partial_{\perp i}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right) + \frac{1}{\mathcal{H}}\left(\frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)}\right)'\Delta\ln a^{(1)} \\
&+ \left[\frac{1}{2}\hat{g}_\mu^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{00}^{(1)} - E_{0\parallel}^{(1)} + \partial_{\bar{\chi}}\left(\Delta x^{0(1)} + \Delta x_{\parallel}^{(1)}\right) + \frac{2}{\bar{\chi}}\Delta x_{\parallel}^{(1)} - 2\kappa^{(1)}\right]\Delta\ln a^{(1)}.
\end{aligned} \tag{53}$$

### III. THE OBSERVED OVER-DENSITY WITH MAGNIFICATION BIAS

Using the results obtained above, in this section we analyze in detail  $n_g$  and, finally, we present the galaxy number over-density up to second order with all relativistic effects that arise from observing on the past lightcone and as function of the magnification bias (see also [26]).

Expanding,  $n_g(x^\mu, \mathcal{M})$ , we find

$$\begin{aligned}
n_g(x^\alpha, \mathcal{M}) &= n_g(\bar{x}^0 + \Delta x^0, \bar{\mathcal{M}} + \Delta\mathcal{M})^{(0)} + n_g(\bar{x}^\alpha + \Delta x^\alpha, \bar{\mathcal{M}} + \Delta\mathcal{M})^{(1)} + \frac{1}{2}n_g(\bar{x}^\alpha, \bar{\mathcal{M}})^{(2)} \\
&= \bar{n}_g + \frac{\partial\bar{n}_g}{\partial\mathcal{M}}\Delta\mathcal{M}^{(1)} + \frac{1}{2}\frac{\partial\bar{n}_g}{\partial\mathcal{M}}\Delta\mathcal{M}^{(2)} + \frac{1}{2}\frac{\partial^2\bar{n}_g}{\partial\mathcal{M}^2}\left(\Delta\mathcal{M}^{(1)}\right)^2 + \frac{\partial\bar{n}_g}{\partial\bar{x}^0}\Delta x^{0(1)} + \frac{\partial^2\bar{n}_g}{\partial\mathcal{M}\partial\bar{x}^0}\Delta x^{0(1)}\Delta\mathcal{M}^{(1)} \\
&+ \frac{1}{2}\frac{\partial\bar{n}_g}{\partial\bar{x}^0}\Delta\bar{x}^{0(2)} + \frac{1}{2}\frac{\partial^2\bar{n}_g}{\partial\bar{x}^{02}}\left(\Delta x^{0(1)}\right)^2 + n_g^{(1)} + \frac{\partial n_g^{(1)}}{\partial\mathcal{M}}\Delta\mathcal{M}^{(1)} + \frac{\partial n_g^{(1)}}{\partial\bar{x}^0}\Delta x^{0(1)} + \frac{\partial n_g^{(1)}}{\partial\bar{x}^i}\Delta x^{i(1)} + \frac{1}{2}n_g^{(2)},
\end{aligned} \tag{54}$$

where  $\bar{n}_g = n_g(\bar{x}^0, \bar{L})^{(0)}$  is the background number density of sources with luminosity exceeding  $\bar{L}$  and, from the second line,  $n_g^{(1)} = n_g(\bar{x}^\alpha, \bar{L})^{(1)}$  and  $n_g^{(2)} = n_g(\bar{x}^\alpha, \bar{L})^{(2)}$ . Now, we define

$$\delta_g^{(1)} = \delta_g(\bar{x}^\alpha, \bar{L})^{(1)} = \frac{n_g(\bar{x}^\alpha, \bar{L})^{(1)}}{n_g(\bar{x}^0, \bar{L})^{(0)}} \quad \text{and} \quad \delta_g^{(2)} = \delta_g(\bar{x}^\alpha, \bar{L})^{(2)} = \frac{n_g(\bar{x}^\alpha, \bar{L})^{(2)}}{n_g(\bar{x}^0, \bar{L})^{(0)}}. \tag{55}$$

Knowing that  $\bar{a} = a(\bar{x}^0)$ ,

$$\frac{\partial\bar{n}_g}{\partial\bar{x}^0}(\bar{x}^0, \bar{L}) = \frac{\partial\bar{n}_g}{\partial\bar{x}^0}\bigg|_{\bar{L}} = \mathcal{H}\frac{\partial\bar{n}_g}{\partial\ln\bar{a}}\bigg|_{\bar{L}} = \mathcal{H}\frac{\partial\bar{n}_g}{\partial\ln\bar{a}}(\bar{a}, \bar{L}), \tag{56}$$



$\Delta\mathcal{M}^{(1)} = -\Delta(\mathcal{M}^{-1})^{(1)}$  and  $\Delta\mathcal{M}^{(2)} = -\Delta(\mathcal{M}^{-1})^{(2)} + 2[\Delta(\mathcal{M}^{-1})^{(1)}]^2$ , we find

$$n_g(x^\alpha, \mathcal{M}) = \bar{n}_g(\bar{x}^0, \bar{L}) + \Delta n_g(\bar{x}^\alpha, \bar{L})^{(1)} + \frac{1}{2}\Delta n_g(\bar{x}^\alpha, \bar{L})^{(2)} \quad (57)$$

where

$$\begin{aligned} \frac{\Delta n_g(\bar{x}^\alpha, \bar{L})^{(1)}}{\bar{n}_g} &= \delta_g^{(1)} + \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{a}} \Delta \ln a^{(1)} - \mathcal{Q} \Delta(\mathcal{M}^{-1})^{(1)}, \\ \frac{\Delta n_g(\bar{x}^\alpha, \bar{L})^{(2)}}{\bar{n}_g} &= \delta_g^{(2)} + \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{a}} \Delta \ln a^{(2)} - \mathcal{Q} \Delta(\mathcal{M}^{-1})^{(2)} + \left(2\mathcal{Q} + \mathcal{Q}^2 - \frac{\partial \mathcal{Q}}{\partial \bar{L}}\right) \left[\Delta(\mathcal{M}^{-1})^{(1)}\right]^2 \\ &\quad + 2\partial_{\parallel} \delta_g^{(1)} \Delta x_{\parallel}^{(1)} + 2\partial_{\perp i} \delta_g^{(1)} \Delta x_{\perp}^{i(1)} - 2\mathcal{Q} \delta_g^{(1)} \Delta(\mathcal{M}^{-1})^{(1)} - 2\mathcal{Q}^{(1)} \Delta(\mathcal{M}^{-1})^{(1)} \\ &\quad + 2 \left[ - \left( \mathcal{Q} \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{a}} + \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right) \Delta(\mathcal{M}^{-1})^{(1)} + \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{a}} \delta_g^{(1)} \right] \Delta \ln a^{(1)} + \frac{2}{\mathcal{H}} \delta_g^{(1)'} \Delta \ln a^{(1)} \\ &\quad + \left[ - \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{a}} + \left( \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{a}} \right)^2 + \frac{\partial^2 \ln \bar{n}_g}{\partial \ln \bar{a}^2} \right] \left( \Delta \ln a^{(1)} \right)^2. \end{aligned} \quad (58)$$

Here we have defined the background and the first order magnification bias

$$\mathcal{Q}(\bar{x}^0, \bar{L}) = \frac{\partial \ln \bar{n}_g}{\partial \mathcal{M}} \Big|_{\bar{a}} = - \frac{\partial \ln \bar{n}_g}{\partial \ln \bar{L}} \Big|_{\bar{a}} \quad \text{and} \quad \mathcal{Q}^{(1)}(\bar{x}^\alpha, \bar{L}) = \frac{\partial \delta_g^{(1)}}{\partial \mathcal{M}} \Big|_{\bar{a}} = - \frac{\partial \delta_g^{(1)}}{\partial \ln \bar{L}} \Big|_{\bar{a}}. \quad (60)$$

Let us point out that usually  $(\partial \mathcal{Q} / \partial \ln \bar{a})$  and  $\mathcal{Q}^{(1)}$  are not generally considered in the literature (for example see [26]). Then, from Eqs. (22), (23), (24), (27), (28), (36) and (57), we obtain the observed fractional number over-density

$$\Delta_g = \frac{n_g(\bar{x}^0, \bar{\mathbf{x}}, \bar{L}) - \bar{n}_g(\bar{x}^0, \bar{L})}{\bar{n}_g(\bar{x}^0, \bar{L})} = \Delta_g^{(1)} + \frac{1}{2} \Delta_g^{(2)}, \quad (61)$$

where<sup>5</sup>

$$\begin{aligned} \Delta_g^{(1)} &= \frac{\Delta n_g^{(1)}}{\bar{n}_g} + 3\Delta \ln a^{(1)} + \Delta \sqrt{-g}^{(1)} + \Delta V^{(1)} = \delta_g^{(1)} + \frac{(1 - \mathcal{Q})}{2} \hat{g}_\mu^{\mu(1)} + (b_e - 2\mathcal{Q}) \Delta \ln a^{(1)} + \partial_{\parallel} \Delta x_{\parallel}^{(1)} \\ &\quad + \frac{2(1 - \mathcal{Q})}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2(1 - \mathcal{Q}) \kappa^{(1)} + (1 - \mathcal{Q}) \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) - \mathcal{Q} \left( E_{\hat{0}\hat{0}}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right), \end{aligned} \quad (63)$$

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<sup>5</sup> Here we have used the following relation

$$\partial_{\bar{\chi}} \left( \Delta x^{0(1)} + \Delta x_{\parallel}^{(1)} \right) = \frac{d}{d\bar{\chi}} \left[ \delta x^{0(1)}(\bar{\chi}) + \delta x_{\parallel}^{(1)}(\bar{\chi}) \right] = \delta \nu^{(1)} + \delta n_{\parallel}^{(1)}. \quad (62)$$

and

$$\begin{aligned}
\Delta_g^{(2)} &= \frac{\Delta n_g^{(2)}}{\bar{n}_g} + 3\Delta \ln a^{(2)} + \Delta \sqrt{-g}^{(2)} + \Delta V^{(2)} + 6 \left( \frac{\Delta n_g^{(1)}}{\bar{n}_g} + \Delta \sqrt{-g}^{(1)} + \Delta V^{(1)} \right) \Delta \ln a^{(1)} + 6 \left( \Delta \ln a^{(1)} \right)^2 \\
&+ 2 \frac{\Delta n_g^{(1)}}{\bar{n}_g} \Delta V^{(1)} + 2 \Delta \sqrt{-g}^{(1)} \Delta V^{(1)} + 2 \frac{\Delta n_g^{(1)}}{\bar{n}_g} \Delta \sqrt{-g}^{(1)} \\
&= \delta^{(2)} + b_e \Delta \ln a^{(2)} + \Delta \sqrt{-g}^{(2)} + \Delta V^{(2)} - \mathcal{Q} \Delta (\mathcal{M}^{-1})^{(2)} + 2 \left[ b_e \delta^{(1)} + \frac{1}{\mathcal{H}} \delta^{(1)'} + b_e \Delta V^{(1)} + b_e \Delta \sqrt{-g}^{(1)} \right. \\
&- \left( \mathcal{Q} b_e + \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right) \Delta (\mathcal{M}^{-1})^{(1)} \left. \right] \Delta \ln a^{(1)} + \left( -b_e + b_e^2 + \frac{\partial b_e}{\partial \ln \bar{a}} \right) \left( \Delta \ln a^{(1)} \right)^2 - 2 \mathcal{Q}^{(1)} \Delta (\mathcal{M}^{-1})^{(1)} \\
&+ \left( 2 \mathcal{Q} + \mathcal{Q}^2 - \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \left[ \Delta (\mathcal{M}^{-1})^{(1)} \right]^2 - 2 \mathcal{Q} \delta^{(1)} \Delta (\mathcal{M}^{-1}) + 2 \partial_{\parallel} \delta^{(1)} \Delta x_{\parallel}^{(1)} + 2 \partial_{\perp i} \delta_g^{(1)} \Delta x_{\perp}^{i(1)} \\
&- 2 \mathcal{Q} \Delta (\mathcal{M}^{-1})^{(1)} \Delta V^{(1)} + 2 \delta^{(1)} \Delta V^{(1)} - 2 \mathcal{Q} \Delta \sqrt{-g}^{(1)} \Delta (\mathcal{M}^{-1})^{(1)} + 2 \Delta \sqrt{-g}^{(1)} \Delta V^{(1)} + 2 \delta^{(1)} \Delta \sqrt{-g}^{(1)} \\
&= \delta_g^{(2)} + \frac{(1 - \mathcal{Q})}{2} \hat{g}_{\mu}^{\mu(2)} + (b_e - 2 \mathcal{Q}) \Delta \ln a^{(2)} + \partial_{\parallel} \Delta x_{\parallel}^{(2)} + \frac{2(1 - \mathcal{Q})}{\bar{\chi}} \Delta x_{\parallel}^{(2)} - 2(1 - \mathcal{Q}) \kappa^{(2)} \\
&+ (1 - \mathcal{Q}) \left( E_{\hat{0}}^{0(2)} + E_{\hat{0}}^{\parallel(2)} \right) - \mathcal{Q} \left( E_{\hat{0}0}^{(2)} - E_{\hat{0}\parallel}^{(2)} \right) - \mathcal{Q} \partial_{\bar{\chi}} \left( \Delta x_{\parallel}^{0(2)} + \Delta x_{\parallel}^{(2)} \right) + \frac{2}{\mathcal{H}} \delta_g^{(1)'} \Delta \ln a^{(1)} + 2 \partial_{\parallel} \delta_g^{(1)} \Delta x_{\parallel}^{(1)} \\
&+ 2 \partial_{\perp}^i \delta_g^{(1)} \Delta x_{\perp i}^{(1)} + 2 \delta_g^{(1)} \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} + \left( 1 - \mathcal{Q} + \mathcal{Q}^2 - \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \left[ \frac{1}{4} \hat{g}_{\mu}^{\mu(1)} \hat{g}_{\nu}^{\nu(1)} + \frac{4}{\bar{\chi}} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\parallel}^{(1)} \right. \\
&- 4 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \kappa^{(1)} + \hat{g}_{\mu}^{\mu(1)} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \hat{g}_{\mu}^{\mu(1)} - 2 \hat{g}_{\mu}^{\mu(1)} \kappa^{(1)} \left. \right] + (1 - \mathcal{Q}) \left[ -\frac{1}{2} \hat{g}_{\mu}^{\nu(1)} \hat{g}_{\nu}^{\mu(1)} \right. \\
&+ \frac{1}{\mathcal{H}} \hat{g}_{\mu}^{\mu(1)'} \Delta \ln a^{(1)} + \left( \partial_{\parallel} \hat{g}_{\mu}^{\mu(1)} \right) \Delta x_{\parallel}^{(1)} + \left( \partial_{\perp i} \hat{g}_{\mu}^{\mu(1)} \right) \Delta x_{\perp}^{i(1)} - 4 \kappa^{(1)} \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} + \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2 |\gamma^{(1)}|^2 \\
&+ \partial_{ij}^{(1)} v^{ij(1)} + \frac{2}{\bar{\chi}} \Delta x_{\perp i}^{(1)} \left( \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} \right) - 2 \left( \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} \right) \left( \partial_{\perp i} \Delta x_{\parallel}^{(1)} \right) + \frac{2}{\mathcal{H}} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right)' \Delta \ln a^{(1)} \\
&+ 2 \partial_{\parallel} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\parallel}^{(1)} + 2 \partial_{\perp i} \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \Delta x_{\perp}^{i(1)} - 2 E_{\hat{0}}^{\parallel(1)} \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) + 2 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} \\
&+ \hat{g}_{\mu}^{\mu(1)} \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} + \hat{g}_{\mu}^{\mu(1)} \delta_g^{(1)} - 2 E_{\hat{0}}^{\perp i(1)} \partial_{\perp i} \left( \Delta x_{\parallel}^{0(1)} + \Delta x_{\parallel}^{(1)} \right) + 2 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \delta_g^{(1)} + \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \delta_g^{(1)} - 4 \delta_g^{(1)} \kappa^{(1)} \left. \right] \\
&+ \left( 1 - \mathcal{Q} + 2 \mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \left[ 2 \left( \kappa^{(1)} \right)^2 + \frac{2}{\bar{\chi}^2} \left( \Delta x_{\parallel}^{(1)} \right)^2 - \frac{4}{\bar{\chi}} \kappa^{(1)} \Delta x_{\parallel}^{(1)} \right] \\
&+ 2 \left( -\mathcal{Q} + \mathcal{Q}^2 - \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) + \left( \mathcal{Q}^2 - \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \left[ 2 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) \right. \\
&+ 2 \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right) \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) - 4 \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \kappa^{(1)} - 4 \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) \kappa^{(1)} + \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \\
&+ \frac{4}{\bar{\chi}} \Delta x_{\parallel}^{(1)} \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) + \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right)^2 + \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right)^2 + \left( E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} \right)^2 + \hat{g}_{\mu}^{\mu(1)} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \\
&+ \hat{g}_{\mu}^{\mu(1)} \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) \left. \right] + \mathcal{Q} \left[ -\frac{2}{\mathcal{H}} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right)' \Delta \ln a^{(1)} - 2 \partial_{\parallel} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\parallel}^{(1)} \right. \\
&- 2 \partial_{\perp i} \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \Delta x_{\perp}^{i(1)} + 2 E_{\hat{0}\perp i}^{(1)} \left( -\frac{1}{\bar{\chi}} \Delta x_{\perp}^{i(1)} + \partial_{\bar{\chi}} \Delta x_{\perp}^{i(1)} \right) + 2 E_{\hat{0}0}^{(1)} \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) - 2 \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} \right) \delta_g^{(1)} \\
&- 2 \left( \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) \delta_g^{(1)} \left. \right] + \left\{ 2 \left[ b_e (1 - \mathcal{Q}) + 2 \mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} - \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right] \left( \frac{1}{2} \hat{g}_{\mu}^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2 \kappa^{(1)} \right) \right. \\
&+ 2 \left[ \mathcal{Q} (2 - b_e) + 2 \mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} - \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right] \left( E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} \right) + 2 (b_e - 2 \mathcal{Q}) \left( \delta_g^{(1)} + \partial_{\bar{\chi}} \Delta x_{\parallel}^{(1)} \right) \\
&- 4 \mathcal{Q}^{(1)} \left. \right\} \Delta \ln a^{(1)} + \left( -b_e + b_e^2 + \frac{\partial b_e}{\partial \ln \bar{a}} + 6 \mathcal{Q} - 4 \mathcal{Q} b_e + 4 \mathcal{Q}^2 - 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} - 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right) \left( \Delta \ln a^{(1)} \right)^2 \\
&- 2 \mathcal{Q}^{(1)} \left( \frac{1}{2} \hat{g}_{\mu}^{\mu(1)} + E_{\hat{0}}^{0(1)} + E_{\hat{0}}^{\parallel(1)} + E_{\hat{0}0}^{(1)} - E_{\hat{0}\parallel}^{(1)} + \delta \nu^{(1)} + \delta n_{\parallel}^{(1)} + \frac{2}{\bar{\chi}} \Delta x_{\parallel}^{(1)} - 2 \kappa^{(1)} \right). \tag{64}
\end{aligned}$$

Here

$$b_e = \frac{\partial \ln \bar{n}_g(\bar{a}, \bar{L})}{\partial \ln \bar{a}} + 3 \quad (65)$$

is the evolution bias term related to the comoving number density. Before concluding this section let us add the following comments:

- In this case,  $b_e$  in Eq. (65) is defined with partial derivatives w.r.t.  $\bar{a}$ . Generally, at linear order,  $b_e$  is defined with the total derivatives instead of the partial one because the redshift distribution of  $n_g$  is usually defined in terms of a fixed threshold  $\bar{L}$ . At second order, to correctly obtain all the terms we can not use this approach.

Instead, if we apply the total derivatives we find the following relation

$$b_{eQ} = \frac{d \ln [\bar{a}^3 \bar{n}_g(\bar{a}, \bar{L}(\bar{a}))]}{d \ln \bar{a}} = b_e - \frac{\partial \ln \bar{L}}{\partial \ln \bar{a}} Q = b_e + 2 \left( 1 + \frac{1}{\bar{\chi} \mathcal{H}} \right) Q, \quad (66)$$

where for last step we have used<sup>6</sup>  $\bar{L} = 4\pi \bar{\mathcal{F}} \bar{\mathcal{D}}_L^2(\bar{a}) = 4\pi \bar{\mathcal{F}} \bar{\chi}^2 / \bar{a}^2$ .

- Obviously, the perturbative expansion made in Eq. (54) is correct if and only if

$$\frac{\partial}{\partial \mathcal{M}} \left( \frac{\partial \bar{n}_g}{\partial \bar{x}^0} \right) = \frac{\partial}{\partial \bar{x}^0} \left( \frac{\partial \bar{n}_g}{\partial \mathcal{M}} \right) \quad \text{or, equivalently,} \quad \frac{\partial b_e}{\partial \ln \bar{L}} = \frac{\partial Q}{\partial \ln \bar{a}}. \quad (67)$$

We think that this consistency equation is very important not only for a theoretical point of view but also for feedback in the data analysis of survey catalogs.

- For simplicity, assuming no velocity bias between galaxy and matter, we can rewrite the magnification bias at first order in the following way

$$Q^{(1)}(\bar{x}^\alpha, \bar{L}) = -\frac{\partial \delta_g^{(1)}}{\partial \ln \bar{L}} = -\frac{\partial \delta_{gCO}^{(1)}}{\partial \ln \bar{L}} = -\frac{\partial b_1(\bar{x}^0, \bar{L})}{\partial \ln \bar{L}} \delta_{mCO}^{(1)}, \quad (68)$$

where  $\delta_{gCO}^{(1)}$  and  $\delta_{mCO}^{(1)}$  are galaxy and cold dark matter over-density in the comoving-time orthogonal (CO) gauge<sup>7</sup> (see also Ref. [25]) respectively. As a result the terms in Eq. (100) proportional to  $Q^{(1)}$  can not be neglected because the bias  $b_1$  depends on the redshift and the luminosity  $\bar{L}$ .

- Taking into account only the following terms of Eq. (100)

$$-2(1-Q)\kappa^{(2)} - 4(1-Q)\delta_g^{(1)}\kappa^{(1)} + 2\left(1-Q+2Q^2-2\frac{\partial Q}{\partial \ln \bar{L}}\right)\left(\kappa^{(1)}\right)^2 - 2(1-Q)|\gamma^{(1)}|^2 + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} \quad (69)$$

we can compare our result with the second order contribution obtained in Eq. (14) of [11] via the standard approach

$$2c_1\delta_g\kappa + c_2\kappa^2 + c_1\gamma^2, \quad (70)$$

where  $c_1 = -2(1-Q)$  and  $c_2 = 2-6Q+4Q^2$ . Approximating for simplicity that  $\gamma^{(1)} \sim \gamma$  and  $\kappa^{(1)} \sim \kappa$ , we note immediately that  $\kappa^{(2)}$  and  $\vartheta_{ij}^{(1)}\vartheta^{ij(1)}$  are usually omitted from standard analyses. Moreover, comparing  $\kappa^2$  terms, we see that the coefficient of  $Q$  are different [in Eq. (69) it is  $-2$  and in Eq. (70) it is  $-6$ ]. This discrepancy is related to the term

$$-2Q\Delta(\mathcal{M}^{-1})^{(1)}\Delta V^{(1)}$$

that we find in the intermediate step of Eq. (100). Finally, there is no  $\partial Q / \partial \ln \bar{L}$  part in Eq. (70).

- Using the relations obtained in Sec. III of [25], we can obtain, in a complete general way, the magnification, the luminosity distance and the observed over-density in a general gauge both at first and second order.

<sup>6</sup> For simplicity, in Eq. (66), we have assumed that  $\bar{\mathcal{F}}$  does not depend explicitly on time and/or if we are considering the bolometric relation between the flux density and the (bolometric) Luminosity.

<sup>7</sup> Let us point out that the comoving-time orthogonal gauge becomes the usual comoving-synchronous gauge when the perturbations are dominated by pressure-free matter, for example in the  $\Lambda$ CDM model.

#### IV. PERTURBATION TERMS IN THE POISSON GAUGE FOR A CONCORDANCE MODEL

We present the observed galaxy number over-density up to second order in redshift space on cosmological scales for a  $\Lambda$ CDM model (without the magnification, see [24]). The standard assumption at first order is that galaxy velocity equals CDM velocity on large scales, and we are assuming that it is reasonable to extend this assumption to second order, since we are dealing only with large scales (i.e. well above the nonlinear scale). Moreover, we assume a concordance background and at first order we neglect anisotropic stress, vector and tensor perturbations.

In the Poisson gauge, the metric and peculiar velocity are

$$ds^2 = a(\eta)^2 \left\{ - \left( 1 + 2\Phi + \Phi^{(2)} \right) d\eta^2 + 2\omega_i^{(2)} d\eta dx^i + \left[ \delta_{ij} \left( 1 - 2\Phi - \Psi^{(2)} \right) + \frac{1}{2} \hat{h}_{ij}^{(2)} \right] dx^i dx^j \right\}, \quad (71)$$

$$v^i = \partial^i v + \frac{1}{2} v^{i(2)}, \quad v^{i(2)} = \partial^i v^{(2)} + \hat{v}^{i(2)}, \quad (72)$$

where we omit the superscript (1) on familiar quantities such as  $\Phi$  and  $\partial^i v$ . At second order, the first-order scalars generate vector perturbations  $\omega_i^{(2)}, \hat{v}^{i(2)}$  and a tensor perturbation  $\hat{h}_{ij}^{(2)}$ .

Now, taking into account the relations obtained in appendix A or in Sec. IV of Ref. [25], for Eqs. (9), (11), (16), (17), (18), (31), (32) we find at first order

$$\Delta \ln a^{(1)} = (\Phi_o - v_{||o}) - \Phi + \partial_{||} v + 2I^{(1)}, \quad (73)$$

$$\delta\chi^{(1)} = - \left( \bar{\chi} + \frac{1}{\mathcal{H}} \right) (\Phi_o - v_{||o}) + \frac{1}{\mathcal{H}} (\Phi - \partial_{||} v) - T^{(1)} - 2 \left( \bar{\chi} + \frac{1}{\mathcal{H}} \right) I^{(1)} - 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi', \quad (74)$$

$$\Delta x^{0(1)} = \frac{1}{\mathcal{H}} \Delta \ln a^{(1)}, \quad (75)$$

$$\Delta x_{||}^{(1)} = -T^{(1)} - \frac{1}{\mathcal{H}} \Delta \ln a^{(1)}, \quad (76)$$

$$\Delta x_{\perp}^{i(1)} = -\bar{\chi} v_{\perp o}^i + 2\bar{\chi} S_{\perp}^{i(1)} - \bar{\chi} \partial_{\perp}^i T^{(1)}, \quad (77)$$

$$\kappa^{(1)} = -v_{||o} + \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \frac{\tilde{\chi}}{\bar{\chi}} \tilde{\nabla}_{\perp}^2 \Phi, \quad (78)$$

$$\gamma_{ij}^{(1)} = -\mathcal{P}_{ij} v_{||o}^{(1)} - n_{(j} v_{\perp i)o}^{(1)} - 2 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ (\bar{\chi} - \tilde{\chi}) \frac{\tilde{\chi}}{\bar{\chi}} \tilde{\partial}_{\perp(i} \tilde{\partial}_{\perp j)} \Phi \right] - \mathcal{P}_{ij} \kappa^{(1)}. \quad (79)$$

Here<sup>8</sup> [24]

$$I^{(1)} = - \int_0^{\bar{\chi}} d\tilde{\chi} \Phi', \quad S^{i(1)} = - \int_0^{\bar{\chi}} d\tilde{\chi} \left( \tilde{\partial}^i \Phi - \frac{1}{\bar{\chi}} n^i \Phi \right), \quad (80)$$

where  $I^{(1)}$  is the integrated Sachs-Wolfe (ISW) effect at first order,

$$S_{\perp}^{i(1)} = \mathcal{P}_j^i S_{\perp}^{j(1)} = - \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\partial}_{\perp}^i \Phi, \quad S_{||}^{(1)} = n_i S^{i(1)} = \Phi_o - \Phi + I^{(1)} + \int_0^{\bar{\chi}} d\tilde{\chi} \frac{\Phi}{\bar{\chi}}. \quad (81)$$

and

$$T^{(1)} = -2 \int_0^{\bar{\chi}} d\tilde{\chi} \Phi \quad (82)$$

is a radial displacement at first order and corresponds to the usual (Shapiro) time delay (STD) term [20]. Another useful relation at first order is the following

$$\partial_{||} \Delta x_{||}^{(1)} = 2\Phi - \partial_{||} v - \frac{1}{\mathcal{H}} \partial_{||}^2 v + \frac{1}{\mathcal{H}} \Phi' - \frac{\mathcal{H}'}{\mathcal{H}^2} \Delta \ln a^{(1)}. \quad (83)$$

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<sup>8</sup> Note that  $\tilde{\partial}_i = \partial/\partial \tilde{x}^i$ .

Instead, at second order, Eqs. (10), (19), (20), (21), (31) turn out (see also [24])

$$\begin{aligned}
\Delta \ln a^{(2)} = & -\Phi^{(2)} + \partial_{\parallel} v^{(2)} + \hat{v}_{\parallel}^{(2)} + 3\Phi^2 - (\partial_{\parallel} v)^2 + \partial_{\perp i} v \partial_{\perp}^i v - 2\partial_{\parallel} v \Phi - \frac{2}{\mathcal{H}} (\Phi - \partial_{\parallel} v) (\Phi' - \partial_{\parallel}^2 v) - 4 \left( \Phi + \frac{1}{\mathcal{H}} \partial_{\parallel}^2 v \right. \\
& \left. - \frac{1}{\mathcal{H}} \Phi' \right) I^{(1)} + 2 \left( 2\Phi' + \partial_{\parallel} \Phi - \partial_{\parallel}^2 v \right) T^{(1)} + 4\bar{\chi} \partial_{\perp i} (-\Phi + \partial_{\parallel} v) S_{\perp}^{i(1)} + 2 \left[ \bar{\chi} \partial_{\perp i} (\Phi - \partial_{\parallel} v) + \partial_{\perp i} v \right] \partial_{\perp}^i T^{(1)} + 8\Phi \kappa^{(1)} \\
& + 2I^{(2)} + 8 \left( I^{(1)} \right)^2 + 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ \Phi'' T^{(1)} + 2\Phi \Phi' + 2\Phi' I^{(1)} + 2\Phi \tilde{\partial}_{\perp j} S_{\perp}^{j(1)} - 2\tilde{\chi} \tilde{\partial}_{\perp i} \Phi' S_{\perp}^{i(1)} - 2 \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right. \\
& \left. + \tilde{\chi} \tilde{\partial}_{\perp i} \Phi' \partial_{\perp}^i T^{(1)} \right] + \Phi_o^{(2)} - v_{\parallel o}^{(2)} - \Phi_o^2 + 8\Phi_o v_{\parallel o} + v_{k o} v_o^k + 2 (\Phi_o - v_{\parallel o}) \left( -\Phi - \frac{1}{\mathcal{H}} \partial_{\parallel}^2 v + \frac{1}{\mathcal{H}} \Phi' + 2I^{(1)} \right) \\
& + 8v_{\parallel o} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi + 2\bar{\chi} v_{\perp o}^{i(1)} \left( \partial_{\perp i} \Phi - \partial_{\perp i} \partial_{\parallel} v - 2\partial_{\perp i} I^{(1)} \right), \tag{84}
\end{aligned}$$

$$\begin{aligned}
\Delta x^{0(2)} = & -\frac{1}{\mathcal{H}} \Phi^{(2)} + \frac{1}{\mathcal{H}} \partial_{\parallel} v^{(2)} + \frac{1}{\mathcal{H}} \hat{v}_{\parallel}^{(2)} + \frac{2}{\mathcal{H}} I^{(2)} - \left( \frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{2}{\mathcal{H}} \right) \Phi^2 - \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{2}{\mathcal{H}} \right) (\partial_{\parallel} v)^2 + \frac{1}{\mathcal{H}} \partial_{\perp i} v \partial_{\perp}^i v + 2 \frac{\mathcal{H}'}{\mathcal{H}^3} \Phi \partial_{\parallel} v \\
& + \frac{2}{\mathcal{H}^2} (\Phi - \partial_{\parallel} v) (-\Phi' + \partial_{\parallel}^2 v) + 4 \left\{ \frac{\mathcal{H}'}{\mathcal{H}^3} \Phi - \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \partial_{\parallel} v - \frac{1}{\mathcal{H}^2} (-\Phi' + \partial_{\parallel}^2 v) - \left( \frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{1}{\mathcal{H}} \right) I^{(1)} \right\} I^{(1)} \\
& + 4 \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} (-\Phi + \partial_{\parallel} v) S_{\perp}^{i(1)} + \frac{2}{\mathcal{H}} \left( 2\Phi' + \partial_{\parallel} \Phi - \partial_{\parallel}^2 v \right) T^{(1)} + \frac{8}{\mathcal{H}} \Phi \kappa^{(1)} + \frac{2}{\mathcal{H}} [\bar{\chi} \partial_{\perp i} (\Phi - \partial_{\parallel} v) + \partial_{\perp i} v] \partial_{\perp}^i T^{(1)} \\
& + \frac{4}{\mathcal{H}} \int_0^{\bar{\chi}} d\tilde{\chi} \left[ \Phi'' T^{(1)} + 2\Phi \Phi' + 2\Phi' I^{(1)} + 2\Phi \tilde{\partial}_{\perp j} S_{\perp}^{j(1)} - 2 \left( \frac{d}{d\tilde{\chi}} \Phi - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} - 2\tilde{\chi} \tilde{\partial}_{\perp i} \Phi' S_{\perp}^{i(1)} + \tilde{\chi} \tilde{\partial}_{\perp i} \Phi' \partial_{\perp}^i T^{(1)} \right] \\
& + \frac{1}{\mathcal{H}} \Phi_o^{(2)} - \frac{1}{\mathcal{H}} v_{\parallel o}^{(2)} - \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{2}{\mathcal{H}} \right) \Phi_o^2 + 2 \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{5}{\mathcal{H}} \right) \Phi_o v_{\parallel o} - \frac{\mathcal{H}'}{\mathcal{H}^3} v_{\parallel o}^2 + \frac{1}{\mathcal{H}} v_{\perp i o} v_{\perp o}^i + \frac{8}{\mathcal{H}} v_{\parallel o} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi \\
& + 2 (\Phi_o - v_{\parallel o}) \left[ \frac{\mathcal{H}'}{\mathcal{H}^3} \Phi - \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \partial_{\parallel} v - \frac{1}{\mathcal{H}^2} \partial_{\parallel}^2 v + \frac{1}{\mathcal{H}^2} \Phi' - 2 \frac{\mathcal{H}'}{\mathcal{H}^3} I^{(1)} \right] + 2 \frac{\bar{\chi}}{\mathcal{H}} v_{\perp o}^i \left( \partial_{\perp i} \Phi - \partial_{\perp i} \partial_{\parallel} v - 2\partial_{\perp i} I^{(1)} \right), \tag{85}
\end{aligned}$$

$$\begin{aligned}
\Delta x_{\parallel}^{(2)} = & -\frac{1}{\mathcal{H}} \Delta \ln a^{(2)} - T^{(2)} - \frac{4}{\mathcal{H}} \Phi \Delta \ln a^{(1)} + \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \left( \Delta \ln a^{(1)} \right)^2 - 4\Phi T^{(1)} - 4\bar{\chi} S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} \\
& + 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left( -\Phi^2 - \Phi' T^{(1)} - 2\Phi \kappa^{(1)} - \tilde{\chi} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} \right) + 8 \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d}{d\tilde{\chi}} \Phi - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \\
& - \bar{\chi} (8\Phi_o v_{\parallel o} + v_{\perp k o} v_{\perp o}^k) - 4v_{\parallel o} \left( T^{(1)} + 2\bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi \right), \tag{86}
\end{aligned}$$

$$\begin{aligned}
\Delta x_{\perp}^{i(2)} = & \int_0^{\bar{\chi}} d\tilde{\chi} \left( 2\omega_{\perp}^{i(2)} - n^j \hat{h}_{jk}^{(2)} \mathcal{P}^{ki} - 8\tilde{\chi} \tilde{\partial}_{\perp}^i \Phi I^{(1)} - 8\tilde{\partial}_{\perp}^i \Phi \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' + 4\Phi \tilde{\partial}_{\perp}^i T^{(1)} \right) \\
& + \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left\{ - \left[ \tilde{\partial}_{\perp}^i \left( \Phi^{(2)} + 2\omega_{\parallel}^{(2)} + \Psi^{(2)} - \frac{1}{2} \hat{h}_{\parallel}^{(2)} \right) + \frac{1}{\tilde{\chi}} \left( -2\omega_{\perp}^{i(2)} + n^k \hat{h}_{kj}^{(2)} \mathcal{P}^{ij} \right) \right] \right. \\
& \left. + 8\Phi \tilde{\partial}_{\perp}^i \Phi - 8\tilde{\partial}_{\perp}^i \Phi I^{(1)} - 4\tilde{\partial}_{\perp}^i \Phi' T^{(1)} + \frac{4}{\tilde{\chi}} \Phi \partial_{\perp}^i T^{(1)} + 4 \left( \tilde{\chi} \mathcal{P}^{im} \tilde{\partial}_{\perp j} \tilde{\partial}_{\perp m} \Phi + \mathcal{P}_j^i \Phi' \right) \left( -2S_{\perp}^{j(1)} + \partial_{\perp}^j T^{(1)} \right) \right\} \\
& - 4 \left( T^{(1)} + \frac{1}{\mathcal{H}} \Delta \ln a^{(1)} + 2\bar{\chi} I^{(1)} + 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) S_{\perp}^{i(1)} + \bar{\chi} \left[ -2\omega_{\perp o}^{i(2)} - v_{\perp o}^{i(2)} + \frac{1}{2} n^j \hat{h}_{jk o}^{(2)} \mathcal{P}^{ki} + 2\Phi_o v_{\perp o}^i - v_{\parallel o} v_{\perp o}^i \right] \\
& + 2v_{\perp o}^j \left[ \mathcal{P}_j^i T^{(1)} + \frac{1}{\mathcal{H}} \mathcal{P}_j^i \Delta \ln a^{(1)} + \bar{\chi}^2 \mathcal{P}^{im} \partial_{\perp j} \partial_{\perp m} \left( T^{(1)} + 2\bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi \right) \right] \tag{87}
\end{aligned}$$

and

$$\begin{aligned}
\kappa^{(2)} = & \frac{1}{2} \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \frac{\tilde{\chi}}{\bar{\chi}} \tilde{\nabla}_\perp^2 \left( \Phi^{(2)} + 2\omega_\parallel^{(2)} + \Psi^{(2)} - \frac{1}{2} \hat{h}_\parallel^{(2)} \right) + \frac{1}{2} \int_0^{\bar{\chi}} d\tilde{\chi} \left( -2\tilde{\partial}_\perp^i \omega_i^{(2)} + \frac{4}{\bar{\chi}} \omega_\parallel^{(2)} + \mathcal{P}^{ij} n^k \tilde{\partial}_i \hat{h}_{jk}^{(2)} \right. \\
& - \frac{3}{\bar{\chi}} \hat{h}_\parallel^{(2)} \left. \right) - 2 \left( 2I^{(1)} + \frac{2}{\bar{\chi}} \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' + \frac{1}{\bar{\chi}} T^{(1)} + \frac{1}{\bar{\chi} \mathcal{H}} \Delta \ln a \right) \left( \kappa^{(1)} - \frac{\bar{\chi}}{2} \nabla_\perp^2 T^{(1)} \right) - 2S_\perp^i \left[ -\partial_{\perp i} T^{(1)} - \frac{1}{\mathcal{H}} \partial_{\perp i} \Delta \ln a^{(1)} \right. \\
& - 2 \left( \bar{\chi} \partial_{\perp i} I^{(1)} + \partial_{\perp i} \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) \left. \right] + 2 \int_0^{\bar{\chi}} d\tilde{\chi} \frac{\tilde{\chi}}{\bar{\chi}} \left[ \frac{4}{\bar{\chi}} \Phi S_\parallel^{(1)} - 2\Phi \tilde{\partial}_{\perp m} S^{m(1)} + 2\tilde{\chi} \tilde{\nabla}_\perp^2 \Phi I^{(1)} + 2\tilde{\nabla}_\perp^2 \Phi \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right. \\
& + 2\tilde{\chi} \partial_{\perp i} \Phi \tilde{\partial}_\perp^i I^{(1)} + 2\partial_{\perp i} \Phi \tilde{\partial}_\perp^i \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' - \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} - \frac{2}{\bar{\chi}} \Phi \kappa^{(1)} \left. \right] + 2 \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \frac{\tilde{\chi}}{\bar{\chi}} \left[ -2\tilde{\partial}_{\perp i} \Phi \tilde{\partial}_\perp^i \Phi \right. \\
& + 2\tilde{\partial}_\perp^i \Phi \tilde{\partial}_{\perp i} I^{(1)} - 2\Phi \tilde{\nabla}_\perp^2 \Phi - \tilde{\nabla}_\perp^2 \Phi T^{(1)} - \tilde{\partial}_\perp^i \Phi \tilde{\partial}_{\perp i} T^{(1)} + 2I^{(1)} \tilde{\nabla}_\perp^2 \Phi + \frac{2}{\bar{\chi}} \left( -\frac{1}{\bar{\chi}} \Phi + \frac{d}{d\bar{\chi}} \Phi \right) \kappa^{(1)} + \frac{2}{\bar{\chi}} \tilde{\partial}_{\perp i} \Phi S_\perp^{i(1)} \\
& - \frac{2}{\bar{\chi}} \Phi \tilde{\partial}_{\perp m} S_\perp^{m(1)} - \frac{2}{\bar{\chi}} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} + \left( \tilde{\partial}_{\perp i} \tilde{\nabla}_\perp^2 \Phi + \frac{1}{\bar{\chi}} \tilde{\partial}_{\perp i} \Phi' \right) \left( 2\tilde{\chi} S_\perp^{i(1)} - \tilde{\chi} \partial_{\perp}^i T^{(1)} \right) - \left( \tilde{\partial}_\perp^{(j} \tilde{\partial}_\perp^{m)} \Phi + \frac{1}{\bar{\chi}} \mathcal{P}^{jm} \Phi' \right. \\
& + \frac{1}{\bar{\chi}^2} \mathcal{P}^{jm} \Phi \left. \right) \left( \gamma_{mj}^{(1)} + \mathcal{P}_{mj} \kappa^{(1)} \right) - \frac{1}{\bar{\chi}} n^{[j} \partial_\perp^{m]} \Phi \theta_{mj}^{(1)} \left. \right] - 2\omega_\parallel^{(2)} - v_\parallel^{(2)} + \frac{3}{4} \hat{h}_\parallel^{(2)} + \frac{1}{2} v_{\perp i o} v_{\perp i o} + 2\Phi_o v_{\parallel o} - 2v_{\parallel o}^2 \\
& - 4v_{\parallel o} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}} \Phi - v_{\perp o}^i \left[ -2S_\perp^{(1)} + 2\partial_{\perp i} T^{(1)} + \bar{\chi}^2 \partial_{\perp i} \nabla_\perp^2 \left( T^{(1)} + 2\bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}} \Phi \right) + \frac{1}{\mathcal{H}} \partial_{\perp i} \Delta \ln a^{(1)} \right] \\
& = \frac{1}{2} \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \frac{\tilde{\chi}}{\bar{\chi}} \tilde{\nabla}_\perp^2 \left( \Phi^{(2)} + 2\omega_\parallel^{(2)} + \Psi^{(2)} - \frac{1}{2} \hat{h}_\parallel^{(2)} \right) + \frac{1}{2} \int_0^{\bar{\chi}} d\tilde{\chi} \left( -2\tilde{\partial}_\perp^i \omega_i^{(2)} + \frac{4}{\bar{\chi}} \omega_\parallel^{(2)} + \mathcal{P}^{ij} n^k \tilde{\partial}_i \hat{h}_{jk}^{(2)} - \frac{3}{\bar{\chi}} \hat{h}_\parallel^{(2)} \right) \\
& - \frac{2}{\bar{\chi}} \left( 2\bar{\chi} I^{(1)} + 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' + T^{(1)} + \frac{1}{\mathcal{H}} \Delta \ln a \right) \left( \kappa^{(1)} - \frac{\bar{\chi}}{2} \nabla_\perp^2 T^{(1)} \right) + 2S_\perp^i \left( \partial_{\perp i} T^{(1)} + \frac{1}{\mathcal{H}} \partial_{\perp i} \Delta \ln a^{(1)} + 2\bar{\chi} \partial_{\perp i} I^{(1)} \right. \\
& + 2\partial_{\perp i} \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \left. \right) + 2 \int_0^{\bar{\chi}} d\tilde{\chi} \frac{\tilde{\chi}}{\bar{\chi}} \left[ +2\tilde{\chi} \tilde{\nabla}_\perp^2 \Phi I^{(1)} + 2\tilde{\nabla}_\perp^2 \Phi \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' + 2\tilde{\chi} \partial_{\perp i} \Phi \tilde{\partial}_\perp^i I^{(1)} + 2\partial_{\perp i} \Phi \tilde{\partial}_\perp^i \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' + \frac{4}{\bar{\chi}} \Phi S_\parallel^{(1)} \right. \\
& - 2\Phi \tilde{\partial}_{\perp m} S^{m(1)} - \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} - \frac{2}{\bar{\chi}} \Phi \kappa^{(1)} \left. \right] + 2 \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \frac{\tilde{\chi}}{\bar{\chi}} \left[ -2\tilde{\partial}_\perp^i \Phi \tilde{\partial}_{\perp i} \Phi + 2\tilde{\partial}_\perp^i \Phi \tilde{\partial}_{\perp i} I^{(1)} - 2\Phi \tilde{\nabla}_\perp^2 \Phi + 2\tilde{\nabla}_\perp^2 \Phi I^{(1)} \right. \\
& - \tilde{\nabla}_\perp^2 \Phi T^{(1)} - \tilde{\partial}_\perp^i \Phi \tilde{\partial}_{\perp i} T^{(1)} + \frac{2}{\bar{\chi}} \left( -\frac{1}{\bar{\chi}} \Phi + \frac{d}{d\bar{\chi}} \Phi \right) \kappa^{(1)} + \frac{1}{\bar{\chi}} \tilde{\partial}_{\perp i} \Phi S_\perp^{i(1)} - \frac{3}{2\bar{\chi}} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} + \tilde{\chi} \left( \tilde{\partial}_{\perp i} \tilde{\nabla}_\perp^2 \Phi + \frac{1}{\bar{\chi}} \tilde{\partial}_{\perp i} \Phi' \right) \\
& \times \left( 2S_\perp^{i(1)} - \partial_{\perp}^i T^{(1)} \right) + \tilde{\chi} \tilde{\partial}_\perp^{(j} \tilde{\partial}_\perp^{m)} \Phi \left( 2\partial_{\perp(m} S_{\perp j)}^{(1)} - \partial_{\perp(m} \partial_{\perp j)} T^{(1)} \right) + 2\Phi' \tilde{\partial}_{\perp m} S_\perp^{m(1)} - \left( \Phi' + \frac{1}{\bar{\chi}} \Phi \right) \tilde{\nabla}_\perp^2 T^{(1)} \left. \right] \\
& - 2\omega_\parallel^{(2)} - v_\parallel^{(2)} + \frac{3}{4} \hat{h}_\parallel^{(2)} + 2\Phi_o v_{\parallel o} + \frac{1}{2} v_{\perp i o} v_{\perp i o} + v_{\parallel o}^2 + \frac{2}{\bar{\chi}} v_{\parallel o} \left( +\bar{\chi} \kappa^{(1)} + T^{(1)} - 2\bar{\chi} I^{(1)} - 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) \\
& + v_{\perp o}^i \left[ 2S_\perp^{i(1)} - 2\partial_{\perp}^i T^{(1)} - \bar{\chi}^2 \partial_{\perp i} \nabla_\perp^2 \left( T^{(1)} + 2\bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}} \Phi \right) - \frac{1}{\mathcal{H}} \partial_{\perp i} \Delta \ln a^{(1)} \right] . \tag{88}
\end{aligned}$$

Here

$$T^{(2)} = - \int_0^{\bar{\chi}} d\tilde{\chi} \left( \Phi^{(2)} + 2\omega_\parallel^{(2)} + \Psi^{(2)} - \frac{1}{2} \hat{h}_\parallel^{(2)} \right) , \tag{89}$$

and

$$I^{(2)} = -\frac{1}{2} \int_0^{\bar{\chi}} d\tilde{\chi} \left( \Phi^{(2)'} + 2\omega_\parallel^{(2)'} + \Psi^{(2)'} - \frac{1}{2} \hat{h}_\parallel^{(2)'} \right) . \tag{90}$$

Using Eqs. (33) and (A6) for  $\Delta x_{\parallel}^{(2)}$ , we find

$$\begin{aligned}
\partial_{\parallel} \Delta x_{\parallel}^{(2)} = & +\Phi^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat{h}_{\parallel}^{(2)} + \frac{1}{\mathcal{H}}\Psi^{(2)'} - \frac{1}{2\mathcal{H}}\hat{h}_{\parallel}^{(2)'} - \frac{1}{\mathcal{H}}\partial_{\parallel}^2 v^{(2)} - \frac{1}{\mathcal{H}}\partial_{\parallel}\hat{v}_{\parallel}^{(2)} - \partial_{\parallel}v^{(2)} - \hat{v}_{\parallel}^{(2)} - \frac{\mathcal{H}'}{\mathcal{H}^2}\Delta \ln a^{(2)} \\
& + \frac{2}{\mathcal{H}^2}(\partial_{\parallel}^2 v)^2 + 4\Phi^2 + 2(\partial_{\parallel}v)^2 + \frac{4}{\bar{\chi}\mathcal{H}}(\partial_{\parallel}v)^2 + \frac{2}{\mathcal{H}^2}(\Phi')^2 - 2\Phi\partial_{\parallel}v + \frac{2}{\mathcal{H}^2}\partial_{\parallel}v\partial_{\parallel}^2\Phi + \frac{6}{\mathcal{H}}\Phi\Phi' + 2\frac{\mathcal{H}'}{\mathcal{H}^3}\Phi\Phi' + \frac{4}{\mathcal{H}}\partial_{\parallel}v\partial_{\parallel}\Phi \\
& - \frac{8}{\mathcal{H}}\Phi\partial_{\parallel}^2v - \frac{2}{\mathcal{H}^2}\Phi\partial_{\parallel}^3v - \frac{2}{\mathcal{H}}\Phi\partial_{\parallel}\Phi + \frac{2}{\mathcal{H}^2}\Phi\frac{d}{d\bar{\chi}}\Phi' - \frac{2}{\mathcal{H}^2}\partial_{\parallel}v\frac{d}{d\bar{\chi}}\Phi' - 2\frac{\mathcal{H}'}{\mathcal{H}^3}\Phi\partial_{\parallel}^2v + \frac{8}{\mathcal{H}}\partial_{\parallel}v\partial_{\parallel}^2v + 2\frac{\mathcal{H}'}{\mathcal{H}^3}\partial_{\parallel}v\partial_{\parallel}^2v - \frac{2}{\mathcal{H}^2}\Phi\partial_{\parallel}^2\Phi \\
& - 2\frac{\mathcal{H}'}{\mathcal{H}^3}\partial_{\parallel}v\Phi' - \frac{4}{\mathcal{H}^2}\partial_{\parallel}^2v\Phi' + \frac{2}{\mathcal{H}}\partial_{\perp i}v\partial_{\perp}^i\Phi - \frac{2}{\mathcal{H}}\partial_{\perp i}v\partial_{\perp}^i\partial_{\parallel}v + \frac{2}{\bar{\chi}\mathcal{H}}\partial_{\perp i}v\partial_{\perp}^i v - 2\partial_{\perp i}v\partial_{\perp}^i v + \frac{2}{\mathcal{H}^2}\partial_{\parallel}v\partial_{\parallel}^3v + \frac{2}{\mathcal{H}}\partial_{\parallel}v\nabla_{\perp}^2v \\
& + 2\left[\frac{1}{\mathcal{H}}\partial_{\parallel}^3v + \partial_{\parallel}^2v - \frac{1}{\mathcal{H}}\partial_{\parallel}\Phi' - 2\partial_{\parallel}\Phi\right]T^{(1)} + 4\bar{\chi}\left(\frac{1}{\mathcal{H}}\partial_{\perp i}\Phi' - \frac{1}{\mathcal{H}}\partial_{\perp i}\partial_{\parallel}^2v - \partial_{\perp i}\partial_{\parallel}v + 2\partial_{\perp i}\Phi\right)S_{\perp}^{i(1)} - 4S_{\perp}^{i(1)}S_{\perp}^{j(1)}\delta_{ij} \\
& + 4\left(\frac{\mathcal{H}'}{\mathcal{H}^3}\partial_{\parallel}^2v - \frac{\mathcal{H}'}{\mathcal{H}^3}\Phi' + \frac{1}{\mathcal{H}}\Phi' + \frac{1}{\mathcal{H}}\partial_{\parallel}^2v + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^2\Phi + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^3v + \frac{1}{\mathcal{H}}\partial_{\parallel}\Phi - \frac{1}{\mathcal{H}^2}\frac{d}{d\bar{\chi}}\Phi'\right)I^{(1)} + 2\left[-\left(1 - \frac{2}{\bar{\chi}\mathcal{H}}\right)\partial_{\perp i}v\right. \\
& \left.+ \bar{\chi}\left(1 - \frac{2}{\bar{\chi}\mathcal{H}}\right)\partial_{\perp i}\partial_{\parallel}v - 2\bar{\chi}\partial_{\perp i}\Phi - \frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}\Phi' + \frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}\partial_{\parallel}^2v\right]\partial_{\perp}^iT^{(1)} - 8\Phi\kappa^{(1)} + 2\left[-2\frac{\mathcal{H}'}{\mathcal{H}^2}\Phi - \frac{2}{\mathcal{H}}\frac{d}{d\bar{\chi}}\Phi\right. \\
& \left.- \left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\left(-\partial_{\parallel}^2v - \mathcal{H}\partial_{\parallel}v + \Phi'\right)\right]\Delta \ln a^{(1)} + \left[-\frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right](\Delta \ln a^{(1)})^2 \\
& + 8\int_0^{\bar{\chi}}d\tilde{\chi}\left[-\Phi\tilde{\partial}_{\perp m}S_{\perp}^{m(1)} + \left(\frac{d}{d\tilde{\chi}}\Phi - \frac{1}{\tilde{\chi}}\Phi\right)\kappa^{(1)}\right] - 8\Phi_o v_{\parallel o} - v_{\perp k o}v_{\perp o}^k\Phi - 8v_{\parallel o}\int_0^{\bar{\chi}}\frac{d\tilde{\chi}}{\tilde{\chi}}\Phi \\
& + 2(\Phi_o - v_{\parallel o})\left[\left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\partial_{\parallel}^2v + \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right)\Phi' + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^2\Phi + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^3v + \frac{1}{\mathcal{H}}\partial_{\parallel}\Phi - \frac{1}{\mathcal{H}^2}\frac{d}{d\bar{\chi}}\Phi'\right] \\
& - 2v_{\perp o}^i\left[-\frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}\left(-\Phi' + \partial_{\parallel}^2v + \mathcal{H}\partial_{\parallel}v\right) - 2S_{\perp}^{j(1)}\delta_{ij} + 2\bar{\chi}\partial_{\perp i}\Phi\right]. \tag{91}
\end{aligned}$$

At this point, using these results, we can focus on the main results of this paper. For  $\mathcal{M}^{-1}$ , from Eqs. (46) and (47), we find

$$\Delta(\mathcal{M}^{-1})^{(1)} = -2\Phi + 2\left(1 - \frac{1}{\bar{\chi}\mathcal{H}}\right)\Delta \ln a^{(1)} - \frac{2}{\bar{\chi}}T^{(1)} - 2\kappa^{(1)}, \tag{92}$$

and

$$\begin{aligned}
\Delta(\mathcal{M}^{-1})^{(2)} = & -2\Psi^{(2)} - \frac{1}{2}\hat{h}_{\parallel}^{(2)} + 2\left(1 - \frac{1}{\bar{\chi}\mathcal{H}}\right)\Delta \ln a^{(2)} - \frac{2}{\bar{\chi}}T^{(2)} - 2\kappa^{(2)} + 4\partial_{\parallel}\Phi T^{(1)} + \frac{2}{\bar{\chi}^2}(T^{(1)})^2 + \frac{4}{\bar{\chi}}\kappa^{(1)}T^{(1)} \\
& + 2(\kappa^{(1)})^2 - 2|\gamma^{(1)}|^2 + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} - 8\bar{\chi}\partial_{\perp i}\Phi S_{\perp}^{i(1)} + 4\bar{\chi}\partial_{\perp i}\Phi\partial_{\perp}^iT^{(1)} + \frac{4}{\mathcal{H}}S_{\perp}^{i(1)}\partial_{\perp i}\Delta \ln a^{(1)} - 4S_{\perp}^{i(1)}S_{\perp}^{j(1)}\delta_{ij} \\
& + 2\left(1 + \frac{1}{\bar{\chi}}\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\bar{\chi}^2\mathcal{H}^2} - \frac{3}{\bar{\chi}\mathcal{H}}\right)(\Delta \ln a^{(1)})^2 + 8\int_0^{\bar{\chi}}d\tilde{\chi}\left[-\Phi\tilde{\partial}_{\perp m}S_{\perp}^{m(1)} + \left(\frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}}\Phi\right)\kappa^{(1)}\right] \\
& + \frac{8}{\bar{\chi}}\int_0^{\bar{\chi}}d\tilde{\chi}\left\{\left[-\Phi^2 - \Phi'T^{(1)} - 2\Phi\kappa^{(1)} - \tilde{\chi}\tilde{\partial}_{\perp i}\Phi\partial_{\perp}^iT^{(1)}\right] + 2(\bar{\chi} - \tilde{\chi})\left[-\Phi\tilde{\partial}_{\perp m}S_{\perp}^{m(1)} + \left(\frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}}\Phi\right)\kappa^{(1)}\right]\right\} \\
& + 4\left[-2\Phi + \frac{1}{\mathcal{H}}\frac{d\Phi}{d\bar{\chi}} - \frac{2}{\bar{\chi}}T^{(1)} + \frac{1}{\bar{\chi}^2\mathcal{H}}T^{(1)} + \frac{1}{\bar{\chi}\mathcal{H}}\kappa^{(1)} - 2\kappa^{(1)}\right]\Delta \ln a^{(1)} - 24\Phi_o v_{\parallel o} - v_{\perp k o}v_{\perp o}^k \\
& - 8v_{\parallel o}\left(\frac{1}{\bar{\chi}}T^{(1)} + 3\int_0^{\bar{\chi}}\frac{d\tilde{\chi}}{\tilde{\chi}}\Phi\right) + 2v_{\perp o}^i\left[2\bar{\chi}\partial_{\perp}^i\Phi + 2S_{\perp}^i - \frac{1}{\mathcal{H}}\partial_{\perp}^i\Delta \ln a^{(1)}\right], \tag{93}
\end{aligned}$$

where

$$\begin{aligned}
\vartheta_{ij}^{(1)} \vartheta^{ij(1)} &= \frac{1}{2} v_{\perp io}^{(1)} v_{\perp o}^{i(1)} + \frac{2}{\bar{\chi}} v_{\perp io}^{(1)} \int_0^{\bar{\chi}} d\tilde{\chi} \left[ (\bar{\chi} - \tilde{\chi}) \tilde{\partial}_{\perp}^i \Phi \right] + \frac{2}{\bar{\chi}^2} \int_0^{\bar{\chi}} d\tilde{\chi} \left[ (\bar{\chi} - \tilde{\chi}) \tilde{\partial}_{\perp i} \Phi \right] \int_0^{\bar{\chi}} d\tilde{\chi} \left[ (\bar{\chi} - \tilde{\chi}) \tilde{\partial}_{\perp}^i \Phi \right] \\
&= \frac{1}{2} v_{\perp io}^{(1)} v_{\perp o}^{i(1)} - v_{\perp io}^{(1)} \left( 2S_{\perp}^{i(1)} - \partial_{\perp}^i T^{(1)} \right) + \frac{1}{2} \delta_{ij} \left( 2S_{\perp}^{i(1)} - \partial_{\perp}^i T^{(1)} \right) \left( 2S_{\perp}^{j(1)} - \partial_{\perp}^j T^{(1)} \right) \\
&= \frac{1}{2\bar{\chi}^2} \Delta x_{\perp i}^{(1)} \Delta x_{\perp}^{i(1)}.
\end{aligned} \tag{94}$$

From Eqs. (49) and (50), the magnification turns out

$$\Delta \mathcal{M}^{(1)} = 2\Phi - 2 \left( 1 - \frac{1}{\mathcal{H}\bar{\chi}} \right) \Delta \ln a^{(1)} + \frac{2}{\bar{\chi}} T^{(1)} + 2\kappa^{(1)}, \tag{95}$$

$$\begin{aligned}
\Delta \mathcal{M}^{(2)} &= 2\Psi^{(2)} + \frac{1}{2} \hat{h}_{\parallel}^{(2)} - 2\Delta \ln a^{(2)} + \frac{2}{\bar{\chi}\mathcal{H}} \Delta \ln a^{(2)} + \frac{2}{\bar{\chi}} T^{(2)} + 2\kappa^{(2)} + 8\Phi^2 + \frac{16}{\bar{\chi}} \Phi T^{(1)} - 4\partial_{\parallel} \Phi T^{(1)} + \frac{6}{\bar{\chi}^2} \left( T^{(1)} \right)^2 \\
&+ 16\Phi \kappa^{(1)} + \frac{12}{\bar{\chi}} \kappa^{(1)} T^{(1)} + 6 \left( \kappa^{(1)} \right)^2 + 2|\gamma^{(1)}|^2 - \vartheta_{ij}^{(1)} \vartheta^{ij(1)} + 8\bar{\chi} \partial_{\perp i} \Phi S_{\perp}^{i(1)} - 4\bar{\chi} \partial_{\perp i} \Phi \partial_{\perp}^i T^{(1)} - \frac{4}{\mathcal{H}} S_{\perp}^{i(1)} \partial_{\perp i} \Delta \ln a^{(1)} \\
&+ 4S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} + 2 \left( 3 + \frac{3}{\mathcal{H}^2 \bar{\chi}^2} - 5 \frac{1}{\mathcal{H}\bar{\chi}} - \frac{1}{\bar{\chi}} \frac{\mathcal{H}'}{\mathcal{H}^3} \right) \left( \Delta \ln a^{(1)} \right)^2 + 4 \left[ -2 \left( 1 - \frac{2}{\mathcal{H}\bar{\chi}} \right) \Phi - \frac{1}{\mathcal{H}} \frac{d\Phi}{d\bar{\chi}} + \frac{1}{\bar{\chi}} \left( -2 + \frac{3}{\bar{\chi}\mathcal{H}} \right) T^{(1)} \right. \\
&+ \left. \left( -2 + \frac{3}{\bar{\chi}\mathcal{H}} \right) \kappa^{(1)} \right] \Delta \ln a^{(1)} - 8 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \\
&- \frac{8}{\bar{\chi}} \int_0^{\bar{\chi}} d\tilde{\chi} \left\{ - \left[ \Phi^2 + \Phi' T^{(1)} + 2\Phi \kappa^{(1)} + \tilde{\chi} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} \right] + 2(\bar{\chi} - \tilde{\chi}) \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \right\} \\
&+ 24\Phi v_{\parallel o} + v_{\perp ko} v_{\perp o}^k + 8v_{\parallel o} \left( \frac{1}{\bar{\chi}} T^{(1)} + 3 \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}} \Phi \right) + 2v_{\perp io} \left[ -2\bar{\chi} \partial_{\perp}^i \Phi - 2S_{\perp}^i + \frac{1}{\mathcal{H}} \partial_{\perp}^i \Delta \ln a^{(1)} \right]
\end{aligned} \tag{96}$$

and, from Eqs. (52) and (53), the luminosity (or the angular) distance is

$$\frac{\mathcal{D}_L^{(1)}}{\mathcal{D}_L} = \frac{\mathcal{D}_A^{(1)}}{\mathcal{D}_A} = -\Phi + \left( 1 - \frac{1}{\mathcal{H}\bar{\chi}} \right) \Delta \ln a^{(1)} - \frac{1}{\bar{\chi}} T^{(1)} - \kappa^{(1)} \tag{97}$$

$$\begin{aligned}
\frac{\mathcal{D}_L^{(2)}}{\mathcal{D}_L} &= \frac{\mathcal{D}_A^{(2)}}{\mathcal{D}_A} = -\Psi^{(2)} - \frac{1}{4} \hat{h}_{\parallel}^{(2)} + \left( 1 - \frac{1}{\mathcal{H}\bar{\chi}} \right) \Delta \ln a^{(2)} - \frac{1}{\bar{\chi}} T^{(2)} - \kappa^{(2)} - \Phi^2 - \frac{2}{\bar{\chi}} \Phi T^{(1)} + 2\partial_{\parallel} \Phi T^{(1)} - 2\Phi \kappa^{(1)} \\
&- |\gamma^{(1)}|^2 + \frac{1}{2} \vartheta_{ij}^{(1)} \vartheta^{ij(1)} - 4\bar{\chi} \partial_{\perp i} \Phi S_{\perp}^{i(1)} + 2\bar{\chi} \partial_{\perp i} \Phi \partial_{\perp}^i T^{(1)} + \frac{2}{\mathcal{H}} S_{\perp}^{i(1)} \partial_{\perp i} \Delta \ln a^{(1)} - 2S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} + \left( \frac{1}{\bar{\chi}} \frac{\mathcal{H}'}{\mathcal{H}^3} - \frac{1}{\bar{\chi}\mathcal{H}} \right) \\
&\times \left( \Delta \ln a^{(1)} \right)^2 + 2 \left[ -\Phi - \frac{1}{\mathcal{H}\bar{\chi}} \Phi + \frac{1}{\mathcal{H}} \frac{d\Phi}{d\bar{\chi}} - \frac{1}{\bar{\chi}} T^{(1)} - \kappa^{(1)} \right] \Delta \ln a^{(1)} + \frac{4}{\bar{\chi}} \int_0^{\bar{\chi}} d\tilde{\chi} \left\{ \left[ -\Phi^2 - \Phi' T^{(1)} - 2\Phi \kappa^{(1)} \right. \right. \\
&- \left. \left. \tilde{\chi} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} \right] + 2(\bar{\chi} - \tilde{\chi}) \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \right\} + 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \\
&- 12\Phi v_{\parallel o} - \frac{1}{2} v_{\perp ko} v_{\perp o}^k - 4v_{\parallel o} \left( \frac{1}{\bar{\chi}} T^{(1)} + 3 \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}} \Phi \right) + v_{\perp io} \left[ 2\bar{\chi} \partial_{\perp}^i \Phi + 2S_{\perp}^i - \frac{1}{\mathcal{H}} \partial_{\perp}^i \Delta \ln a^{(1)} \right].
\end{aligned} \tag{98}$$

Finally, for the observed over-density, we find<sup>9</sup>

$$\begin{aligned}
\Delta_g^{(1)} &= \delta_g^{(1)} + \left[ b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2 \frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}} \right] \Delta \ln a^{(1)} + (-1 + 2\mathcal{Q}) \Phi - \frac{1}{\mathcal{H}} \partial_{\parallel}^2 v + \frac{1}{\mathcal{H}} \Phi' - 2 \frac{(1-\mathcal{Q})}{\bar{\chi}} T^{(1)} \\
&- 2(1-\mathcal{Q}) \kappa^{(1)},
\end{aligned} \tag{99}$$

<sup>9</sup> When one chooses a suitable gauge, like the Poisson gauge that we use in this section, and setting initial conditions correctly there are no gauge modes because the observed over-density  $\Delta_g$  is gauge invariant.



$$\begin{aligned}
\Delta_g^{(2)} = & \delta_g^{(2)} + \left[ b_e - 2\mathcal{Q} - \frac{\mathcal{H}'}{\mathcal{H}^2} - (1 - \mathcal{Q}) \frac{2}{\bar{\chi}\mathcal{H}} \right] \Delta \ln a^{(2)} - (1 - \mathcal{Q}) \left( +2\Psi^{(2)} + \frac{1}{2}\hat{h}_{\parallel}^{(2)} \right) - (1 - \mathcal{Q}) \frac{2}{\bar{\chi}} T^{(2)} - 2(1 - \mathcal{Q}) \kappa^{(2)} \\
& + \Phi^{(2)} + \frac{1}{\mathcal{H}} \Psi^{(2)'} - \frac{1}{2\mathcal{H}} \hat{h}_{\parallel}^{(2)'} - \frac{1}{\mathcal{H}} \partial_{\parallel}^2 v^{(2)} - \frac{1}{\mathcal{H}} \partial_{\parallel} \hat{v}_{\parallel}^{(2)} + 2(-1 + 2\mathcal{Q}) \Phi \delta_g^{(1)} - \frac{2}{\mathcal{H}} \delta_g^{(1)} \partial_{\parallel}^2 v + \frac{2}{\mathcal{H}} \delta_g^{(1)} \Phi' + \frac{2}{\mathcal{H}} \left( 1 + \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \Phi \Phi' \\
& + \left( -5 + 4\mathcal{Q} + 4\mathcal{Q}^2 - 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \Phi^2 + 2\mathcal{Q} \Phi \partial_{\parallel} v + \left( 1 + 2\mathcal{Q} + \frac{4}{\bar{\chi}\mathcal{H}} \right) (\partial_{\parallel} v)^2 - \frac{2}{\mathcal{H}} \left( 1 + \mathcal{Q} + \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \Phi \partial_{\parallel}^2 v + \frac{2}{\mathcal{H}^2} (\Phi')^2 \\
& + \frac{2}{\mathcal{H}^2} (\partial_{\parallel}^2 v)^2 + \frac{2}{\mathcal{H}^2} \partial_{\parallel} v \partial_{\parallel}^2 \Phi + \frac{4}{\mathcal{H}} \partial_{\parallel} v \partial_{\parallel} \Phi - \frac{2}{\mathcal{H}^2} \Phi \partial_{\parallel}^3 v - \frac{2}{\mathcal{H}} \Phi \partial_{\parallel} \Phi + \frac{2}{\mathcal{H}^2} \Phi \frac{d\Phi'}{d\bar{\chi}} - \frac{2}{\mathcal{H}^2} \partial_{\parallel} v \frac{d\Phi'}{d\bar{\chi}} + \frac{2}{\mathcal{H}} \left( 3 + \mathcal{Q} + \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \partial_{\parallel} v \partial_{\parallel}^2 v \\
& - \frac{2}{\mathcal{H}^2} \Phi \partial_{\parallel}^2 \Phi + \frac{2}{\mathcal{H}} \left( 1 - \mathcal{Q} - \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \partial_{\parallel} v \Phi' - \frac{4}{\mathcal{H}^2} \partial_{\parallel}^2 v \Phi' + \frac{2}{\mathcal{H}} \partial_{\perp i} v \partial_{\perp}^i \Phi - \frac{2}{\mathcal{H}} \partial_{\perp i} v \partial_{\perp}^i \partial_{\parallel} v + \left( -1 + \frac{2}{\bar{\chi}\mathcal{H}} \right) \partial_{\perp i} v \partial_{\perp}^i v \\
& + \frac{2}{\mathcal{H}^2} \partial_{\parallel} v \partial_{\parallel}^3 v + \frac{2}{\mathcal{H}} \partial_{\parallel} v \nabla_{\perp}^2 v + \left\{ \left[ -2b_e - 4\mathcal{Q} + 4b_e \mathcal{Q} - 8\mathcal{Q}^2 + 8 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} + 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} + 2 \frac{\mathcal{H}'}{\mathcal{H}^2} (1 - \mathcal{Q}) + \frac{4}{\bar{\chi}\mathcal{H}} \left( -1 + \mathcal{Q} + 2\mathcal{Q}^2 \right. \right. \right. \\
& \left. \left. \left. - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \right] \Phi + 2 \frac{\mathcal{H}'}{\mathcal{H}^2} \mathcal{Q} \partial_{\parallel} v + 2 \left[ b_e - 2\mathcal{Q} - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar{\chi}\mathcal{H}} (1 - \mathcal{Q}) \right] \delta_g^{(1)} - \frac{2}{\mathcal{H}} \frac{d\delta_g^{(1)}}{d\bar{\chi}} + \frac{2}{\mathcal{H}} \left[ -b_e + 2\mathcal{Q} + \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{2}{\bar{\chi}\mathcal{H}} (1 - \mathcal{Q}) \right] \partial_{\parallel}^2 v \right. \\
& \left. + \frac{2}{\mathcal{H}} \left[ -2 + b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar{\chi}\mathcal{H}} (1 - \mathcal{Q}) \right] \Phi' - \frac{4}{\mathcal{H}} \mathcal{Q} \partial_{\parallel} \Phi + 4 \left[ - \left( b_e - b_e \mathcal{Q} + 2\mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} - \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right) + \frac{\mathcal{H}'}{\mathcal{H}^2} (1 - \mathcal{Q}) \right. \right. \\
& \left. \left. + \frac{1}{\bar{\chi}\mathcal{H}} \left( 1 - \mathcal{Q} + 2\mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \right] \left( \frac{1}{\bar{\chi}} T^{(1)} + \kappa^{(1)} \right) \right\} \Delta \ln a^{(1)} + \left\{ -b_e + b_e^2 + \frac{\partial b_e}{\partial \ln \bar{a}} + 6\mathcal{Q} - 4\mathcal{Q} b_e + 4\mathcal{Q}^2 \right. \\
& \left. - 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} - 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} + (1 - 2b_e + 4\mathcal{Q}) \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + 3 \left( \frac{\mathcal{H}'}{\mathcal{H}^2} \right)^2 + \frac{6}{\bar{\chi}} \frac{\mathcal{H}'}{\mathcal{H}^3} (1 - \mathcal{Q}) + \frac{2}{\bar{\chi}^2 \mathcal{H}^2} \left( 1 - \mathcal{Q} + 2\mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \right. \\
& \left. + \frac{2}{\bar{\chi}\mathcal{H}} \left[ 1 - 2b_e - \mathcal{Q} + 2b_e \mathcal{Q} - 4\mathcal{Q}^2 + 4 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} + 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{a}} \right] \right\} (\Delta \ln a^{(1)})^2 + 4 \left[ + \frac{1}{\mathcal{H}} \left( 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \Phi' + \frac{1}{\mathcal{H}} \partial_{\parallel} \Phi \right. \\
& \left. + \frac{1}{\mathcal{H}} \left( 1 + \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \partial_{\parallel}^2 v + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^2 \Phi + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^3 v - \frac{1}{\mathcal{H}^2} \frac{d\Phi'}{d\bar{\chi}} \right] I^{(1)} + \left[ - \frac{4}{\bar{\chi}} (1 - \mathcal{Q}) \delta_g^{(1)} - 2\partial_{\parallel} \delta_g^{(1)} - \frac{4}{\bar{\chi}\mathcal{H}} (1 - \mathcal{Q}) \Phi' \right. \\
& \left. + \frac{4}{\bar{\chi}} \left( -1 + \mathcal{Q} + 2\mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \Phi + 2(1 - 2\mathcal{Q}) \partial_{\parallel} \Phi + \frac{4}{\bar{\chi}\mathcal{H}} (1 - \mathcal{Q}) \partial_{\parallel}^2 v + \frac{2}{\mathcal{H}} \partial_{\parallel}^3 v - \frac{2}{\mathcal{H}} \partial_{\parallel} \Phi' \right] T^{(1)} + \left( 1 - \mathcal{Q} + 2\mathcal{Q}^2 \right. \\
& \left. - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \left[ \frac{2}{\bar{\chi}^2} (T^{(1)})^2 + \frac{4}{\bar{\chi}} T^{(1)} \kappa^{(1)} \right] + 4 \left[ - \left( 1 - \mathcal{Q} - 2\mathcal{Q}^2 + 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) \Phi + \frac{1}{\mathcal{H}} (1 - \mathcal{Q}) \partial_{\parallel}^2 v - \frac{1}{\mathcal{H}} (1 - \mathcal{Q}) \Phi' \right. \\
& \left. - (1 - \mathcal{Q}) \delta_g^{(1)} \right] \kappa^{(1)} + (1 - \mathcal{Q}) \vartheta_{ij}^{(1)} \vartheta^{ij(1)} + 2 \left( 1 - \mathcal{Q} + 2\mathcal{Q}^2 - 2 \frac{\partial \mathcal{Q}}{\partial \ln \bar{L}} \right) (\kappa^{(1)})^2 - 2(1 - \mathcal{Q}) |\gamma^{(1)}|^2 + 4 \left[ \frac{\bar{\chi}}{\mathcal{H}} \left( \partial_{\perp i} \Phi' \right. \right. \\
& \left. \left. - \partial_{\perp i} \partial_{\parallel}^2 v \right) + \bar{\chi} \partial_{\perp i} \delta_g^{(1)} + \bar{\chi} \partial_{\perp i} \Phi - 2\bar{\chi} (1 - \mathcal{Q}) \partial_{\perp i} \Phi + \frac{1}{\mathcal{H}} (1 - \mathcal{Q}) \partial_{\perp i} \Delta \ln a^{(1)} \right] S_{\perp}^{i(1)} - 4(1 - \mathcal{Q}) S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} \\
& + 2 \left[ \frac{2}{\bar{\chi}\mathcal{H}} \partial_{\perp i} v - \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} \Phi' + \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} \partial_{\parallel}^2 v - \frac{2}{\mathcal{H}} \partial_{\perp i} \partial_{\parallel} v - \bar{\chi} \partial_{\perp i} \delta_g^{(1)} - \bar{\chi} \partial_{\perp i} \Phi + 2\bar{\chi} (1 - \mathcal{Q}) \partial_{\perp i} \Phi \right] \partial_{\perp}^i T^{(1)} \\
& + 4\mathcal{Q}^{(1)} \left[ \Phi - \left( 1 - \frac{1}{\bar{\chi}\mathcal{H}} \right) \Delta \ln a^{(1)} + \frac{1}{\bar{\chi}} T^{(1)} + \kappa^{(1)} \right] + 8(1 - \mathcal{Q}) \left\{ \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \right. \\
& \left. - \frac{1}{\bar{\chi}} \int_0^{\bar{\chi}} d\tilde{\chi} \left( \Phi^2 + \Phi' T^{(1)} + 2\Phi \kappa^{(1)} + \tilde{\chi} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} \right) + \frac{1}{\bar{\chi}} \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left[ -2\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + 2 \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right] \right\} \\
& - (1 - \mathcal{Q}) (24\Phi_o v_{\parallel o} + v_{\perp k o} v_{\perp o}^k) - 8(1 - \mathcal{Q}) v_{\parallel o} \left( \frac{1}{\bar{\chi}} T^{(1)} + 3 \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}} \Phi \right) + 2(\Phi_o - v_{\parallel o}) \left[ \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \partial_{\parallel}^2 v \right. \\
& \left. + \left( -\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \Phi' + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^2 \Phi + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^3 v + \frac{1}{\mathcal{H}} \partial_{\parallel} \Phi - \frac{1}{\mathcal{H}^2} \frac{d\Phi'}{d\bar{\chi}} \right] + v_{\perp o}^i \left[ -2\bar{\chi} \partial_{\perp i} \Phi + 2 \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} (-\Phi' + \partial_{\parallel}^2 v) - 2\bar{\chi} \partial_{\perp i} \delta_g^{(1)} \right] \\
& + (1 - \mathcal{Q}) v_{\perp i o} \left[ 4S_{\perp}^i - \frac{2}{\mathcal{H}} \partial_{\perp}^i \Delta \ln a^{(1)} + 4\bar{\chi} \partial_{\perp}^i \Phi \right]
\end{aligned}$$

Here found several new terms and contributions that we cannot a priori neglect. In particular: *i*) comparing our result with [26], we have new magnification terms proportional to  $(\partial Q/\partial \ln \bar{a})$  and  $Q^{(1)}(\bar{x}^\alpha, \bar{L}) = -(\partial b_1/\partial \ln \bar{L})\delta_{m\text{CO}}^{(1)}$ . *ii*) Comparing  $\kappa^2$  terms in Eqs. (69) or (100) with Eq. (14) of [11], we note that the coefficient of  $Q$  is different [in Eq. (69) or Eq. (100) it is  $-2$  and in Eq. (14) of [11] it is  $-6$ ]. This discrepancy is related to the term  $-2Q\Delta(\mathcal{M}^{-1})^{(1)}\Delta V^{(1)}$  that we find in the intermediate step of Eq. (100). *iii*) There is no  $\partial Q/\partial \ln \bar{L}$  in Eq. (14) of [11]. *iv*) It is clear that in Eq.(100) we generalize the result obtained in Eq. (14) of [11] adding all the relativistic contributions from velocities, Sachs-Wolfe, integrated SW and time-delay terms. *v*) If we set  $Q = Q^{(1)} = 0$  we find the same results that have been obtained in [24], see Eqs. (B1) and (B2) in Appendix B.

Let us conclude this section with the following comment on the correct frame to define the local bias. Fluctuations of galaxy number density are related to the underlying matter density fluctuation  $\delta_m$  on cosmological scales by a local bias. In order to define this correctly, we need to choose an appropriate frame where the baryon velocity perturbation vanishes. Then the baryon rest frame coincides with the CDM rest frame and, in  $\Lambda$ CDM, this rest frame is defined up to second order by the comoving-synchronous gauge (S) [42–47]. In this S-gauge, the galaxy and matter over-densities are gauge invariant [48].

By setting initial conditions correctly, the gauge mode in comoving-synchronous gauge can be removed (since it is a function only on spatial coordinates) and this gauge is equivalent to Lagrangian frame. The correct frame to define the local bias is the Lagrangian frame. Indeed, this frame has the advantage that the local (Lagrangian) bias is related to the halo mass function through the peak-background split approach [43, 49–51].

The S-gauge is defined by the conditions  $g_{00} = -1$ ,  $g_{0i} = 0$  and  $v^i = 0$ . Then

$$ds^2 = a(\eta)^2 \left\{ -d\eta^2 + \left[ \delta_{ij} - 2\psi\delta_{ij} + \left( \partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2 \right) \xi + \frac{1}{2}h_{ij}^{(2)} \right] dx^i dx^j \right\}, \quad (101)$$

where  $h_{ij}^{(2)} = -2\psi^{(2)}\delta_{ij} + F_{ijS}^{(2)}$ , with  $F_{ijS}^{(2)} = (\partial_i\partial_j - \delta_{ij}\nabla^2/3)\xi^{(2)} + \partial_i\hat{\xi}_j^{(2)} + \partial_j\hat{\xi}_i^{(2)} + \hat{h}_{ij}^{(2)}$ ,  $\partial_i\hat{\xi}^{i(2)} = \partial_i\hat{h}^{ij(2)} = 0$ . Here, for simplicity, we neglect vector and tensor perturbations at first order, i.e.  $\xi_j = \hat{h}_{ij} = 0$ .

In order to obtain the galaxy fractional number overdensity  $\delta_{gS}$ , we transform the metric perturbations from the Poisson to comoving-synchronous gauge [24, 42]

$$\delta_g = \delta_{gS} - b_e \mathcal{H}v + 3\mathcal{H}v, \quad (102)$$

$$\begin{aligned} \delta_g^{(2)} = & \delta_{gS}^{(2)} - b_e \mathcal{H}v^{(2)} + 3\mathcal{H}v^{(2)} + \left( b_e \mathcal{H}' - 3\mathcal{H}' + \mathcal{H}^2 \frac{\partial b_e}{\partial \ln \bar{a}} + b_e^2 \mathcal{H}^2 - 6b_e \mathcal{H}^2 + 9\mathcal{H}^2 \right) v^2 + \mathcal{H}b_e vv' - 3\mathcal{H}vv' \\ & - 2\mathcal{H}b_e v \delta_{gS} + 6\mathcal{H}v \delta_{gS} - 2v \delta_{gS}' - \frac{1}{2} \partial^i \xi (-b_e \mathcal{H} \partial_i v + 3\mathcal{H} \partial_i v + 2\partial_i \delta_{gS}) - (b_e - 3) \mathcal{H} \nabla^{-2} \left( v \nabla^2 v' - v' \nabla^2 v \right. \\ & \left. - 6\partial_i \Phi \partial^i v - 6\Phi \nabla^2 v + \frac{1}{2} \partial_i \xi \partial^i \nabla^2 v + \frac{1}{2} \partial_i v \partial^i \nabla^2 \xi + \partial_i \partial_j \xi \partial^i \partial^j v \right). \end{aligned} \quad (103)$$

Note that  $v = \xi'/2$ .

Then the scale-independent bias at first and at second order (down to mildly nonlinear scales) is given by [51, 52]

$$\delta_{gS}^{(1)} + \frac{1}{2} \delta_{gS}^{(2)} = b_1^L \delta_{mS}^{(1)} + \frac{1}{2} b_1^L \delta_{mS}^{(2)} + \frac{1}{2} b_2^L (\delta_{mS}^{(1)})^2. \quad (104)$$

Expressions (102)–(104) can then be substituted into (100), thus incorporating the bias correctly.

Finally, in order to make a correct result, it is important to study the degrees of freedom of these equations have for given initial conditions. It will also be important for an accurate analysis of the ‘contamination’ of primordial non-Gaussianity by relativistic projection effects. This is the subject of ongoing work [41].

## V. CONCLUSIONS

In this paper we have presented the observed galaxy counts to second order in redshift space on cosmological scales for a  $\Lambda$ CDM model, including all general relativistic effects and as function of the magnification. The main result is given by Eq. (100).

We have found new terms and contributions that we cannot neglect:

*Firstly*, comparing our result with [26], we have new magnification terms proportional to

$$(\partial Q/\partial \ln \bar{a}) \quad \text{and} \quad Q^{(1)}(\bar{x}^\alpha, \bar{L}) = -(\partial b_1/\partial \ln \bar{L})\delta_{m\text{CO}}^{(1)}.$$

Secondly, comparing  $\kappa^2$  terms in Eqs. (69) or (100) with Eq. (14) of [11], we note that the coefficient of  $\mathcal{Q}$  is different [in Eq. (69) or Eq. (100) it is  $-2$  and in Eq. (14) of [11] it is  $-6$ ]. This discrepancy is related to the term

$$-2\mathcal{Q}\Delta(\mathcal{M}^{-1})^{(1)}\Delta V^{(1)}$$

that we find in the intermediate step of Eq. (100). Then, there is no  $\partial\mathcal{Q}/\partial\ln\bar{L}$  in Eq. (14) of [11].

Lastly we generalize the result obtained in Eq. (14) of [11] adding all the relativistic contributions from velocities, Sachs-Wolfe, integrated SW and time-delay terms.

The results presented in this work suggest that we have to take into account the magnification corrections when making measurements of non-Gaussianity. If we neglect these effects, we could potentially not estimate correctly the sensitivity of galaxy surveys to primordial non-Gaussianity [41]. Finally, this allows for an investigation of whether general relativistic effects are measurable beyond the linear approximation in the mildly nonlinear regime in future surveys.

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### Appendix A: Useful relations in the Poisson gauge

From Eq. (71) the perturbation of FRW metric  $g_{\mu\nu}$  and  $g^{\mu\nu}$  is

$$\begin{aligned} g_{00} &= -a^2 (1 + 2\Phi + \Phi^{(2)}), & g^{00} &= -a^{-2} [1 - 2\Phi - \Phi^{(2)} + 4\Phi^2], \\ g_{0i} &= a^2 \omega_i^{(2)}, & g^{0i} &= a^{-2} \omega_i^{(2)}, \\ g_{ij} &= a^2 \left( \delta_{ij} - 2\delta_{ij}\Phi - \delta_{ij}\Psi^{(2)} + \hat{h}_{ij}^{(2)}/2 \right), & g^{ij} &= a^{-2} \left[ \delta^{ij} + 2\delta^{ij}\Phi + \delta^{ij}\Psi^{(2)} - \hat{h}^{ij(2)}/2 + 4\delta^{ij}(\Phi)^2 \right], \end{aligned} \quad (\text{A1})$$

For four-velocity  $u^\mu$ , we find

$$u_0 = -a \left[ 1 + \Phi + \frac{1}{2}\Phi^{(2)} - \frac{1}{2}\Phi^2 + \frac{1}{2}v_k v^k \right], \quad (\text{A2})$$

$$u_i = a \left[ v_i + \frac{1}{2} \left( v_i^{(2)} + 2\omega_i^{(2)} \right) - 2\Phi v_i \right], \quad (\text{A3})$$

$$u^0 = \frac{1}{a} \left[ 1 - \Phi - \frac{1}{2}\Phi^{(2)} + \frac{3}{2}\Phi^2 + \frac{1}{2}v_k v^k \right], \quad (\text{A4})$$

$$u^i = \frac{1}{a} \left( v^i + \frac{1}{2}v^{i(2)} \right). \quad (\text{A5})$$

Given  $T_m^{\mu\nu} = \rho_m u^\mu u^\nu$ , i.e. the cold dark matter stress-energy tensor, for first and second-order perturbations we obtain

$$\begin{aligned} \delta_m' + \partial_i v^i - 3\Phi' &= 0, \\ v^{i'} + \mathcal{H}v^i + \partial^i \Phi &= 0, \\ \frac{1}{2}\delta_m^{(2)'} + \frac{1}{2}\partial_i v^{i(2)} - \frac{3}{2}\Psi^{(2)'} + \frac{1}{4}\hat{h}_i^{i(2)'} - \mathcal{H}v^i v_i + (\Phi + \delta_m)\partial_i v^i + v^i \partial_i \delta_m - 3\delta_m \Phi' - 3v^j \partial_j \Phi - 6\Phi \Phi' &= 0, \\ \left( \frac{1}{2}v^{i(2)} + \omega^{i(2)} \right)' + \mathcal{H} \left( \frac{1}{2}v^{i(2)} + \omega^{i(2)} \right) + \frac{1}{2}\partial^i \Phi^{(2)} - v^i \partial_j v^j - 2v^i \Phi' - \Phi \partial^i \Phi + 2\partial^i \Phi \Phi &= 0. \end{aligned} \quad (\text{A6})$$

The geodesic equation for the comoving null geodesic vector  $k^\mu(\chi) = (dx^\mu/d\chi)(\chi)$  is

$$\frac{dk^\mu(\chi)}{d\chi} + \hat{\Gamma}_{\alpha\beta}^\mu(x^\gamma) k^\alpha(\chi) k^\beta(\chi) = 0 \quad (\text{A7})$$

where  $\hat{\Gamma}_{\alpha\beta}^{\mu}$  are the Christoffel symbols defined using the comoving metric  $\hat{g}_{\mu\nu} = g_{\mu\nu}/a^2$  or  $\hat{g}^{\mu\nu} = a^2 g^{\mu\nu}$ . Expanding  $k^{\mu}(\chi)$  and  $\hat{\Gamma}_{\alpha\beta}^{\mu}(x^{\gamma})$  up to second order in the observed coordinates, we have

$$\begin{aligned} k^{\mu}(\chi) &= k^{\mu}(\bar{\chi}) + \delta\chi \frac{dk^{\mu}}{d\bar{\chi}}(\bar{\chi}) + \frac{1}{2}\delta\chi^2 \frac{d^2k^{\mu}}{d\bar{\chi}^2}(\bar{\chi}) , \\ \hat{\Gamma}_{\alpha\beta}^{\mu}(x^{\gamma}) &= \hat{\Gamma}_{\alpha\beta}^{\mu}(\bar{x}^{\gamma}) + \Delta x^{\nu} \frac{\partial}{\partial \bar{x}^{\nu}} \hat{\Gamma}_{\alpha\beta}^{\mu}(\bar{x}^{\gamma}) + \frac{1}{2} \Delta x^{\nu} \Delta x^{\sigma} \frac{\partial^2}{\partial \bar{x}^{\nu} \partial \bar{x}^{\sigma}} \hat{\Gamma}_{\alpha\beta}^{\mu}(\bar{x}^{\gamma}) . \end{aligned} \quad (\text{A8})$$

After some tedious calculations, we can write the geodesic equation for  $k^{\mu}(\bar{\chi})$  in the following way<sup>10</sup>

$$\frac{dk^{\mu}(\bar{\chi})}{d\bar{\chi}} + \left( \hat{\Gamma}_{\alpha\beta}^{\mu}(\bar{x}^{\gamma}) + \delta x^{\sigma} \frac{\partial \hat{\Gamma}_{\alpha\beta}^{\mu}}{\partial \bar{x}^{\sigma}}(\bar{x}^{\gamma}) \right) k^{\alpha}(\bar{\chi}) k^{\beta}(\bar{\chi}) = 0 . \quad (\text{A9})$$

It is clear that we recover the result obtained in [53].

Now, at zeroth order, we obtain Eq. (6). At first order, Eq. (A9) yields

$$\frac{d}{d\bar{\chi}} \left( \delta\nu^{(1)} - 2\Phi \right) = 2\Phi' , \quad \frac{d}{d\bar{\chi}} \left( \delta n^{i(1)} - 2\Phi n^i \right) = -2\partial^i \Phi . \quad (\text{A10})$$

At second order we have

$$\begin{aligned} \frac{d}{d\bar{\chi}} \left( \delta\nu^{(2)} - 2\Phi^{(2)} - 2\omega_{\parallel}^{(2)} + 4\Phi\delta\nu^{(1)} \right) &= \Phi^{(2)'} + 2\omega_{\parallel}^{(2)'} + \Psi^{(2)'} - \frac{1}{2}\hat{h}_{\parallel}^{(2)'} + 4\delta n^i \partial_i \Phi + 4\delta n_{\parallel} \Phi' \\ -4 \left( \frac{d}{d\bar{\chi}} \Phi' + \Phi'' \right) T^{(1)} + 4 \frac{d}{d\bar{\chi}} \left( \frac{d}{d\bar{\chi}} \Phi + \Phi' \right) \delta x_{\parallel}^{(1)} &+ 4 \left[ \partial_{\perp i} \left( \frac{d}{d\bar{\chi}} \Phi + \Phi' \right) - \frac{1}{\bar{\chi}} \partial_{\perp i} \Phi \right] \delta x_{\perp}^{i(1)} \end{aligned} \quad (\text{A11})$$

and

$$\begin{aligned} \frac{d}{d\bar{\chi}} \left( \delta n^{i(2)} - 2\omega^{i(2)} - 2\Psi^{(2)} n^i + \hat{h}_j^{i(2)} n^j - 4\delta n^{i(1)} \Phi \right) &= -\partial^i \Phi^{(2)} - 2\partial^i \omega_{\parallel}^{(2)} + \frac{2}{\bar{\chi}} \omega_{\perp}^{i(2)} - \partial^i \Psi^{(2)} + \frac{1}{2} \partial^i \hat{h}_{\parallel}^{(2)} \\ -\frac{1}{\bar{\chi}} \mathcal{P}^{ij} \hat{h}_{jk}^{(2)} n^k + 4\delta\nu^{(1)} \left( \partial^i \Phi + n^i \Phi' \right) + 4\delta n^{j(1)} \left( -n_j \partial^i \Phi + n^i \partial_j \Phi \right) &+ 4 \left( \partial^i \Phi - n^i \frac{d}{d\bar{\chi}} \Phi \right)' T^{(1)} \\ -4 \frac{d}{d\bar{\chi}} \left( \partial^i \Phi - n^i \frac{d}{d\bar{\chi}} \Phi \right) \delta x_{\parallel}^{(1)} - 4 \left[ \partial_{\perp l} \left( \partial^i \Phi - n^i \frac{d}{d\bar{\chi}} \Phi \right) + \frac{1}{\bar{\chi}} \mathcal{P}_l^j \left( \delta_j^i \frac{d}{d\bar{\chi}} \Phi + n^i \partial_j \Phi \right) \right] &\delta x_{\perp}^{l(1)} . \end{aligned} \quad (\text{A12})$$

To solve Eqs. (A10), (A11) and (A12) we require the values of  $\delta\nu^{(1)}$ ,  $\delta\nu^{(2)}$ ,  $\delta n^{i(1)}$  and  $\delta n^{i(2)}$  today. In this case we need all the components of the comoving tetrad  $E_{\mu}^{\hat{\alpha}}$  which is defined through the following relations

$$\hat{g}^{\mu\nu} E_{\mu}^{\hat{\alpha}} E_{\nu}^{\hat{\beta}} = \eta^{\hat{\alpha}\hat{\beta}} , \quad \eta_{\hat{\alpha}\hat{\beta}} E_{\mu}^{\hat{\alpha}} E_{\nu}^{\hat{\beta}} = \hat{g}_{\mu\nu} , \quad \hat{g}^{\mu\nu} E_{\nu}^{\hat{\beta}} = E^{\hat{\beta}\mu} , \quad \eta_{\hat{\alpha}\hat{\beta}} E_{\nu}^{\hat{\beta}} = E_{\hat{\beta}\nu} , \quad (\text{A13})$$

where  $\eta_{\hat{\alpha}\hat{\beta}}$  the comoving Minkowski metric. If we choose  $u^{\mu}$  as the time-like basis vector, then  $u_{\mu} = a E_{\hat{0}\mu}$  and  $u^{\mu} = a^{-1} E_{\hat{0}}^{\mu}$ . In the background  $E_{\hat{0}\mu}^{(0)} = (-1, \mathbf{0})$ , and, at first and second order, we get

$$\begin{aligned} E_{\hat{0}0} &= -\Phi , & E_{\hat{0}i} &= v_i , \\ E_{\hat{a}0} &= -v_{\hat{a}} , & E_{\hat{a}i} &= -\delta_{\hat{a}i} \Phi , \\ \frac{1}{2} E_{\hat{0}0}^{(2)} &= -\frac{1}{2} \Phi^{(2)} + \frac{1}{2} \Phi^2 - \frac{1}{2} v_k v^k , \\ \frac{1}{2} E_{\hat{0}i}^{(2)} &= \frac{1}{2} \left( v_i^{(2)} + 2\omega_i^{(2)} \right) - 2\Phi v_i , \\ \frac{1}{2} E_{\hat{a}0}^{(2)} &= -\frac{1}{2} v_{\hat{a}}^{(2)} , \\ \frac{1}{2} E_{\hat{a}i}^{(2)} &= -\frac{1}{2} \delta_{\hat{a}j} \Psi^{(2)} + \frac{1}{4} \hat{h}_{\hat{a}j}^{(2)} + \frac{1}{2} v_i v_{\hat{a}} - \frac{1}{2} \delta_{\hat{a}j} \Phi^2 . \end{aligned} \quad (\text{A14})$$

<sup>10</sup> It is valid up to second order.

Assuming  $a_o = 1$  for  $\bar{\chi} = 0$ , we have  $k_{\hat{o}o} = (E_{\hat{o}\mu}k^\mu)|_o = 1$ ,  $k_{\hat{a}o} = (E_{\hat{a}\mu}k^\mu)|_o = n_{\hat{a}}$ . Then we find, at first order,

$$\delta\nu_o^{(1)} = \Phi_o + v_{\parallel o}, \quad \delta n_o^{\hat{a}(1)} = -v_o^{\hat{a}} + n^{\hat{a}}\Phi_o, \quad (\text{A15})$$

and, at second order,

$$\begin{aligned} \delta\nu_o^{(2)} &= \Phi_o^{(2)} + v_{\parallel o}^{(2)} + 2\omega_{\parallel o}^{(2)} - 3(\Phi_o)^2 - 4v_{\parallel o}\Phi_o - v_{ko}v_o^k \\ \delta n_o^{\hat{a}(2)} &= -v_o^{\hat{a}(2)} + n^{\hat{a}}\Psi_o^{(2)} - \frac{1}{2}n^i\hat{h}_{io}^{\hat{a}(2)} + v_o^{\hat{a}}v_{\parallel o} + 3n^{\hat{a}}(\Phi_o)^2, \end{aligned} \quad (\text{A16})$$

where we define  $v_o^i \equiv (\partial^i v)_o$ , i.e.  $\partial^i v$  evaluated at the observer, and  $v_o^{i(2)} \equiv (\partial^i v^{(2)})_o + \hat{v}_o^{i(2)}$ .

From Eq. (A17) and the constraint from Eq. (A15), we obtain at first order

$$\delta\nu^{(1)} = -(\Phi_o - v_{\parallel o}) + 2\Phi + 2 \int_0^{\bar{\chi}} d\tilde{\chi} \Phi' = -(\Phi_o - v_{\parallel o}) + 2\Phi - 2I^{(1)}, \quad (\text{A17})$$

$$\delta n^{i(1)} = -v_o^i - n^i\Phi_o + 2n^i\Phi - 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\partial}^i \Phi = n^i\delta n_{\parallel}^{(1)} + \delta n_{\perp}^{i(1)}, \quad (\text{A18})$$

where

$$\delta n_{\parallel}^{(1)} = \Phi_o - v_{\parallel o} + 2I^{(1)}, \quad \delta n_{\perp}^{i(1)} = -v_{\perp o}^i + 2S_{\perp}^{i(1)}. \quad (\text{A19})$$

Let us point out the following useful relation  $\delta n_{\parallel}^{(1)} + \delta\nu^{(1)} = 2\Phi$ .

At second order we find

$$\begin{aligned} \delta\nu^{(2)} &= -\Phi_o^{(2)} + v_{\parallel o}^{(2)} + (\Phi_o)^2 - 8\Phi_o v_{\parallel o} - v_{ko}v_o^k + 4(\Phi_o - v_{\parallel o})(\Phi - I^{(1)}) - 8v_{\parallel o} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi + 2\Phi^{(2)} + 2\omega_{\parallel}^{(2)} - 8\Phi^2 \\ &+ 8\Phi I^{(1)} - 2I^{(2)} - 8(I^{(1)})^2 - 8\Phi\kappa^{(1)} + 4\left(\frac{d}{d\tilde{\chi}}\Phi + \Phi'\right)\delta x_{\parallel}^{(1)} - 4\Phi'T^{(1)} + 4\partial_{\perp i}\Phi\delta x_{\perp}^{i(1)} \\ &+ 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi''T^{(1)} - 2\Phi'\Phi - 2\Phi'I^{(1)} - 2\Phi\tilde{\partial}_{\perp j}S_{\perp}^{j(1)} + 2\left(\frac{d}{d\tilde{\chi}}\Phi - \frac{1}{\tilde{\chi}}\Phi\right)\kappa^{(1)} + \tilde{\partial}_{\perp i}\Phi'\delta x_{\perp}^{i(1)} \right], \end{aligned} \quad (\text{A20})$$

and, splitting  $\delta n^{i(2)} = n^i\delta n_{\parallel}^{(2)} + \delta n_{\perp}^{i(2)}$ , we obtain

$$\begin{aligned} \delta n_{\parallel}^{(2)} &= \Phi_o^{(2)} - v_{\parallel o}^{(2)} + v_{\parallel o}^2 - \Phi_o^2 + 4(\Phi_o - v_{\parallel o})I^{(1)} + 4v_{\perp o}S_{\perp}^{i(1)} - \Phi^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat{h}_{\parallel}^{(2)} + 2I^{(2)} + 4\Phi^2 - 8(I^{(1)})^2 \\ &- 4\delta_{ij}S_{\perp}^{i(1)}S_{\perp}^{j(1)} + 8 \int_0^{\bar{\chi}} d\tilde{\chi} (\Phi'\Phi) - 4\Phi'\delta x_{\parallel}^{(1)} + 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ \Phi''T^{(1)} + 2\Phi'I^{(1)} - \tilde{\partial}_{\perp i}\Phi'\delta x_{\perp}^{i(1)} \right] \end{aligned} \quad (\text{A21})$$

and

$$\begin{aligned} \delta n_{\perp}^{i(2)} &= -2\omega_{\perp o}^{i(2)} - v_{\perp o}^{i(2)} + \frac{1}{2}n^j\hat{h}_{jk}^{(2)}\mathcal{P}^{ki} + v_{\parallel o}v_{\perp o}^i + 4(\Phi_o - v_{\parallel o})S_{\perp}^{i(1)} - 4v_{\perp o}^i\Phi - 4v_{\perp o} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi + 2\omega_{\perp}^{i(2)} - n^j\hat{h}_{jk}^{(2)}\mathcal{P}^{ki} \\ &+ 2S_{\perp}^{i(2)} + 8\Phi S_{\perp}^{i(1)} - 4\partial_{\perp}^i\Phi\delta x_{\parallel}^{(1)} - \frac{4}{\tilde{\chi}}\Phi\delta x_{\perp}^{i(1)} + 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ 2\Phi\tilde{\partial}_{\perp}^i\Phi - 2\tilde{\partial}_{\perp}^i\Phi I^{(1)} + \tilde{\partial}_{\perp}^i\Phi'T^{(1)} + \frac{2}{\tilde{\chi}}\Phi S_{\perp}^{i(1)} \right. \\ &\left. - \left( \mathcal{P}^{im}\tilde{\partial}_{\perp j}\tilde{\partial}_{\perp m}\Phi + \frac{1}{\tilde{\chi}}\mathcal{P}_j^i\Phi' + \frac{1}{\tilde{\chi}^2}\mathcal{P}_j^i\Phi \right)\delta x_{\perp}^{j(1)} \right], \end{aligned} \quad (\text{A22})$$

where

$$S_{\perp}^{i(2)} = -\frac{1}{2} \int_0^{\bar{\chi}} d\tilde{\chi} \left[ \tilde{\partial}_{\perp}^i \left( \Phi^{(2)} + 2\omega_{\parallel}^{(2)} + \Psi^{(2)} - \frac{1}{2}\hat{h}_{\parallel}^{(2)} \right) + \frac{1}{\tilde{\chi}} \left( -2\omega_{\perp}^{i(2)} + n^k\hat{h}_{kj}^{(2)}\mathcal{P}^{ij} \right) \right]. \quad (\text{A23})$$

Combining Eqs. (A20) and (A21) we obtain following useful relation

$$\begin{aligned}
\delta\nu^{(2)} + \delta n_{\parallel}^{(2)} &= -8\Phi_o v_{\parallel o} - v_{\perp k o} v_{\perp o}^k + 4(\Phi_o - v_{\parallel o})\Phi - 8v_{\parallel o} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi + 4v_{\perp i o} S_{\perp}^{i(1)} + \Phi^{(2)} + 2\omega_{\parallel}^{(2)} + \Psi^{(2)} \\
&- \frac{1}{2}\hat{h}_{\parallel}^{(2)} - 4\Phi^2 + 8\Phi I^{(1)} - 4S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} + 4\frac{d\Phi}{d\tilde{\chi}} \delta x_{\parallel}^{(1)} - 4\Phi' T^{(1)} + 4\partial_{\perp i} \Phi \delta x_{\perp}^{i(1)} - 8\Phi \kappa^{(1)} \\
&+ 8 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right]. \tag{A24}
\end{aligned}$$

From Eqs. (A17) and (A19) we find

$$\delta x^{0(1)} = -\bar{\chi}(\Phi_o - v_{\parallel o}) + 2 \int_0^{\bar{\chi}} d\tilde{\chi} [\Phi + (\bar{\chi} - \tilde{\chi}) \Phi'] = -\bar{\chi}(\Phi_o - v_{\parallel o}) - T^{(1)} - 2\bar{\chi} I^{(1)} - 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \tag{A25}$$

$$\delta x_{\parallel}^{(1)} = \bar{\chi}(\Phi_o - v_{\parallel o}) - 2 \int_0^{\bar{\chi}} d\tilde{\chi} [(\bar{\chi} - \tilde{\chi}) \Phi'] = \bar{\chi}(\Phi_o - v_{\parallel o}) + 2\bar{\chi} I^{(1)} + 2 \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi', \tag{A26}$$

$$\delta x_{\perp}^{i(1)} = -\bar{\chi} v_{\perp o}^i - 2 \int_0^{\bar{\chi}} d\tilde{\chi} [(\bar{\chi} - \tilde{\chi}) \tilde{\partial}_{\perp}^i \Phi] = -\bar{\chi} v_{\perp o}^i + 2\bar{\chi} S_{\perp}^{i(1)} - \bar{\chi} \partial_{\perp}^i T^{(1)}, \tag{A27}$$

to first order.

At second order,

$$\begin{aligned}
\delta x^{0(2)} &= \bar{\chi} \left( -\Phi_o^{(2)} + v_{\parallel o}^{(2)} + \Phi_o^2 - 8\Phi_o v_{\parallel o} - v_{k o} v_o^k \right) + 4\bar{\chi}(\Phi_o - v_{\parallel o})(\Phi - I^{(1)}) - 8v_{\parallel o} \left( \frac{1}{2} T^{(1)} + \bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi \right) \\
&+ 2\bar{\chi} v_{\perp o}^i \left( \partial_{\perp}^i T^{(1)} + 2\bar{\chi} \partial_{\perp i} I^{(1)} + 2\partial_{\perp i} \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) + 8\Phi \left( \bar{\chi} I^{(1)} + \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) - 4\bar{\chi} S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} \\
&+ 2 \int_0^{\bar{\chi}} d\tilde{\chi} \left[ \Phi^{(2)} + \omega_{\parallel}^{(2)} - 4\Phi^2 - 4(I^{(1)})^2 + 4\bar{\chi} \Phi' I^{(1)} + 4\Phi' \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' - 2\Phi' T^{(1)} - 4\Phi \kappa^{(1)} + 2S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} \right. \\
&\left. - 2\bar{\chi} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} \right] + \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left[ \Phi^{(2)'} + 2\omega_{\parallel}^{(2)'} + \Psi^{(2)'} - \frac{1}{2} \hat{h}_{\parallel}^{(2)'} - 8\Phi' \Phi - 4\Phi'' T^{(1)} - 8\Phi' I^{(1)} - 8\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} \right. \\
&\left. + 8 \left( \frac{d}{d\tilde{\chi}} \Phi - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} + 4\bar{\chi} \tilde{\partial}_{\perp i} \Phi' \left( 2S_{\perp}^{i(1)} - \partial_{\perp}^i T^{(1)} \right) \right], \tag{A28}
\end{aligned}$$

$$\begin{aligned}
\delta x_{\parallel}^{(2)} &= \bar{\chi} \left( \Phi_o^{(2)} - v_{\parallel o}^{(2)} + v_{\parallel o}^2 - \Phi_o^2 \right) + 4\bar{\chi}(\Phi_o - v_{\parallel o}) I^{(1)} + 2\bar{\chi} v_{\perp o}^i \left( 2S_{\perp}^{i(1)} - \partial_{\perp}^i T^{(1)} - 2\bar{\chi} \partial_{\perp i} I^{(1)} - 2\partial_{\perp i} \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) \\
&+ \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi^{(2)} + \Psi^{(2)} - \frac{1}{2} \hat{h}_{\parallel}^{(2)} + 4\Phi^2 + 8(I^{(1)})^2 - 4\delta_{ij} S_{\perp}^{i(1)} S_{\perp}^{j(1)} - 8\bar{\chi} \Phi' I^{(1)} - 8\Phi' \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right] \\
&+ \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left[ -\Phi^{(2)'} - 2\omega_{\parallel}^{(2)'} - \Psi^{(2)'} + \frac{1}{2} \hat{h}_{\parallel}^{(2)'} + 8\Phi \Phi' + 4\Phi'' T^{(1)} + 8\Phi' I^{(1)} + 4\bar{\chi} \tilde{\partial}_{\perp i} \Phi' \left( -2S_{\perp}^{i(1)} + \partial_{\perp}^i T^{(1)} \right) \right], \tag{A29}
\end{aligned}$$

and

$$\begin{aligned}
\delta x_{\perp}^{i(2)} &= \bar{\chi} \left( -2\omega_{\perp o}^{i(2)} - v_{\perp o}^{i(2)} + \frac{1}{2} n^j \hat{h}_{jk}^{(2)} \mathcal{P}^{ki} + v_{\parallel o} v_{\perp o}^i \right) + 4\bar{\chi}(\Phi_o - v_{\parallel o}) S_{\perp}^{i(1)} + 2v_{\perp o}^j \left[ -2\mathcal{P}_j^i \left( \bar{\chi} I^{(1)} + \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) \right. \\
&+ \bar{\chi}^2 \mathcal{P}^{im} \partial_{\perp j} \partial_{\perp m} \left( T^{(1)} + 2\bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi \right) \left. \right] + \int_0^{\bar{\chi}} d\tilde{\chi} \left( 2\omega_{\perp}^{i(2)} - n^j \hat{h}_{jk}^{(2)} \mathcal{P}^{ki} - 8\bar{\chi} \tilde{\partial}_{\perp}^i \Phi I^{(1)} - 8\tilde{\partial}_{\perp}^i \Phi \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' + 4\Phi \tilde{\partial}_{\perp}^i T^{(1)} \right) \\
&+ \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left\{ -\left[ \tilde{\partial}_{\perp}^i \left( \Phi^{(2)} + 2\omega_{\parallel}^{(2)} + \Psi^{(2)} - \frac{1}{2} \hat{h}_{\parallel}^{(2)} \right) + \frac{1}{\tilde{\chi}} \left( -2\omega_{\perp}^{i(2)} + n^k \hat{h}_{kj}^{(2)} \mathcal{P}^{ij} \right) \right] + 8\Phi \tilde{\partial}_{\perp}^i \Phi - 8\tilde{\partial}_{\perp}^i \Phi I^{(1)} \right. \\
&\left. - 4\tilde{\partial}_{\perp}^i \Phi' T^{(1)} + \frac{4}{\tilde{\chi}} \Phi \partial_{\perp}^i T^{(1)} + 4 \left( \bar{\chi} \mathcal{P}^{im} \tilde{\partial}_{\perp j} \tilde{\partial}_{\perp m} \Phi + \mathcal{P}_j^i \Phi' \right) \left( -2S_{\perp}^{j(1)} + \partial_{\perp}^j T^{(1)} \right) \right\}. \tag{A30}
\end{aligned}$$

Combining Eqs. (A28) and (A29) we have

$$\begin{aligned}
\delta x^{0(2)} + \delta x_{\parallel}^{(2)} &= -\bar{\chi} (8\Phi_o v_{\parallel o} + v_{\perp k o} v_{\perp o}^k) + 4\bar{\chi} (\Phi_o - v_{\parallel o}) \Phi - 4v_{\parallel o} \left( T^{(1)} + 2\bar{\chi} \int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\tilde{\chi}} \Phi \right) + 4\bar{\chi} v_{\perp o}^i S_{\perp}^{i(1)} - T^{(2)} \\
&+ 8\Phi \left( \bar{\chi} I^{(1)} + \int_0^{\bar{\chi}} d\tilde{\chi} \tilde{\chi} \Phi' \right) - 4\bar{\chi} S_{\perp}^{i(1)} S_{\perp}^{j(1)} \delta_{ij} + 4 \int_0^{\bar{\chi}} d\tilde{\chi} \left( -\Phi^2 - \Phi' T^{(1)} - 2\Phi \kappa^{(1)} - \tilde{\chi} \tilde{\partial}_{\perp i} \Phi \partial_{\perp}^i T^{(1)} \right) \\
&+ 8 \int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left[ -\Phi \tilde{\partial}_{\perp m} S_{\perp}^{m(1)} + \left( \frac{d}{d\tilde{\chi}} \Phi - \frac{1}{\tilde{\chi}} \Phi \right) \kappa^{(1)} \right].
\end{aligned} \tag{A31}$$

### Appendix B: Final result with $Q = 0$ or $Q = 1$

In this Appendix we show two particular cases of Eqs. (99) and (100).

#### 1. $Q = 0, Q^{(1)} = 0$

At first order (this particular solution has been computed previously in [24])

$$\Delta_g^{(1)} = \delta_g^{(1)} + \left( b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar{\chi} \mathcal{H}} \right) \Delta \ln a^{(1)} - \Phi - \frac{1}{\mathcal{H}} \partial_{\parallel}^2 v + \frac{1}{\mathcal{H}} \Phi' - \frac{2}{\bar{\chi}} T^{(1)} - 2\kappa^{(1)}, \tag{B1}$$

and at second order (see also [24])

$$\begin{aligned}
\Delta_g^{(2)} = & \delta_g^{(2)} + \left( b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar{\chi}\mathcal{H}} \right) \Delta \ln a^{(2)} - 2\Psi^{(2)} - \frac{1}{2}\hat{h}_{\parallel}^{(2)} - \frac{2}{\bar{\chi}}T^{(2)} - 2\kappa^{(2)} + \Phi^{(2)} + \frac{1}{\mathcal{H}}\Psi^{(2)'} - \frac{1}{2\mathcal{H}}\hat{h}_{\parallel}^{(2)'} \\
& - \frac{1}{\mathcal{H}}\partial_{\parallel}^2 v^{(2)} - \frac{1}{\mathcal{H}}\partial_{\parallel}\hat{v}_{\parallel}^{(2)} - 2\Phi\delta_g^{(1)} - \frac{2}{\mathcal{H}}\delta_g^{(1)}\partial_{\parallel}^2 v + \frac{2}{\mathcal{H}}\delta_g^{(1)}\Phi' + \frac{2}{\mathcal{H}}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Phi\Phi' - 5\Phi^2 + \left(1 + \frac{4}{\bar{\chi}\mathcal{H}}\right)(\partial_{\parallel}v)^2 \\
& - \frac{2}{\mathcal{H}}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Phi\partial_{\parallel}^2 v + \frac{2}{\mathcal{H}^2}(\Phi')^2 + \frac{2}{\mathcal{H}^2}(\partial_{\parallel}^2 v)^2 + \frac{2}{\mathcal{H}^2}\partial_{\parallel}v\partial_{\parallel}^2\Phi + \frac{4}{\mathcal{H}}\partial_{\parallel}v\partial_{\parallel}\Phi - \frac{2}{\mathcal{H}^2}\Phi\partial_{\parallel}^3 v - \frac{2}{\mathcal{H}}\Phi\partial_{\parallel}\Phi + \frac{2}{\mathcal{H}^2}\Phi\frac{d\Phi'}{d\bar{\chi}} \\
& - \frac{2}{\mathcal{H}^2}\partial_{\parallel}v\frac{d\Phi'}{d\bar{\chi}} + \frac{2}{\mathcal{H}}\left(3 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\partial_{\parallel}v\partial_{\parallel}^2 v - \frac{2}{\mathcal{H}^2}\Phi\partial_{\parallel}^2\Phi + \frac{2}{\mathcal{H}}\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\partial_{\parallel}v\Phi' - \frac{4}{\mathcal{H}^2}\partial_{\parallel}^2 v\Phi' + \frac{2}{\mathcal{H}}\partial_{\perp i}v\partial_{\perp}^i\Phi - \frac{2}{\mathcal{H}}\partial_{\perp i}v\partial_{\perp}^i\partial_{\parallel}v \\
& + \left(-1 + \frac{2}{\bar{\chi}\mathcal{H}}\right)\partial_{\perp i}v\partial_{\perp}^i v + \frac{2}{\mathcal{H}^2}\partial_{\parallel}v\partial_{\parallel}^3 v + \frac{2}{\mathcal{H}}\partial_{\parallel}v\nabla_{\perp}^2 v + \left[ \left(-2b_e + 2\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{4}{\bar{\chi}\mathcal{H}}\right)\Phi + 2\left(b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar{\chi}\mathcal{H}}\right)\delta_g^{(1)} \right. \\
& - \frac{2}{\mathcal{H}}\frac{d\delta_g^{(1)}}{d\bar{\chi}} + \frac{2}{\mathcal{H}}\left(-b_e + \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{2}{\bar{\chi}\mathcal{H}}\right)\partial_{\parallel}^2 v + \frac{2}{\mathcal{H}}\left(-2 + b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar{\chi}\mathcal{H}}\right)\Phi' + 4\left(-\left(b_e - \frac{\partial\mathcal{Q}}{\partial\ln\bar{a}}\right) + \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{1}{\bar{\chi}\mathcal{H}}\right) \\
& \times \left.\left(\frac{1}{\bar{\chi}}T^{(1)} + \kappa^{(1)}\right)\right] \Delta \ln a^{(1)} + \left[ -b_e + b_e^2 + \frac{\partial b_e}{\partial\ln\bar{a}} + (1 - 2b_e)\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{6}{\bar{\chi}}\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{2}{\bar{\chi}^2\mathcal{H}^2} \right. \\
& + \frac{2}{\bar{\chi}\mathcal{H}} \left. \left(\Delta \ln a^{(1)}\right)^2 + 4\left[ + \frac{1}{\mathcal{H}}\left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Phi' + \frac{1}{\mathcal{H}}\partial_{\parallel}\Phi + \frac{1}{\mathcal{H}}\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\partial_{\parallel}^2 v + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^2\Phi + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^3 v - \frac{1}{\mathcal{H}^2}\frac{d\Phi'}{d\bar{\chi}} \right] I^{(1)} \right. \\
& + \left(-\frac{4}{\bar{\chi}}\delta_g^{(1)} - 2\partial_{\parallel}\delta_g^{(1)} - \frac{4}{\bar{\chi}\mathcal{H}}\Phi' - \frac{4}{\bar{\chi}}\Phi + 2\partial_{\parallel}\Phi + \frac{4}{\bar{\chi}\mathcal{H}}\partial_{\parallel}^2 v + \frac{2}{\mathcal{H}}\partial_{\parallel}^3 v - \frac{2}{\mathcal{H}}\partial_{\parallel}\Phi'\right)T^{(1)} + \frac{2}{\bar{\chi}^2}(T^{(1)})^2 + \frac{4}{\bar{\chi}}T^{(1)}\kappa^{(1)} \\
& + 4\left(-\Phi + \frac{1}{\mathcal{H}}\partial_{\parallel}^2 v - \frac{1}{\mathcal{H}}\Phi' - \delta_g^{(1)}\right)\kappa^{(1)} + \vartheta_{ij}^{(1)}\vartheta^{ij(1)} + 2\left(\kappa^{(1)}\right)^2 - 2|\gamma^{(1)}|^2 + 4\left[\frac{\bar{\chi}}{\mathcal{H}}\left(\partial_{\perp i}\Phi' - \partial_{\perp i}\partial_{\parallel}^2 v\right) + \bar{\chi}\partial_{\perp i}\delta_g^{(1)} \right. \\
& + \bar{\chi}\partial_{\perp i}\Phi - 2\bar{\chi}\partial_{\perp i}\Phi + \frac{1}{\mathcal{H}}\partial_{\perp i}\Delta \ln a^{(1)} \left. \right] S_{\perp}^{i(1)} - 4S_{\perp}^{i(1)}S_{\perp}^{j(1)}\delta_{ij} + 2\left(\frac{2}{\bar{\chi}\mathcal{H}}\partial_{\perp i}v - \frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}\Phi' + \frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}\partial_{\parallel}^2 v - \frac{2}{\mathcal{H}}\partial_{\perp i}\partial_{\parallel}v \right. \\
& - \bar{\chi}\partial_{\perp i}\delta_g^{(1)} - \bar{\chi}\partial_{\perp i}\Phi + 2\bar{\chi}\partial_{\perp i}\Phi \left. \right)\partial_{\perp}^i T^{(1)} + 8\left\{ \int_0^{\bar{\chi}} d\tilde{\chi} \left[ -\Phi\tilde{\partial}_{\perp m}S_{\perp}^{m(1)} + \left(\frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}}\Phi\right)\kappa^{(1)} \right] \right. \\
& - \frac{1}{\bar{\chi}}\int_0^{\bar{\chi}} d\tilde{\chi} \left( \Phi^2 + \Phi'T^{(1)} + 2\Phi\kappa^{(1)} + \tilde{\chi}\tilde{\partial}_{\perp i}\Phi\partial_{\perp}^i T^{(1)} \right) + \frac{1}{\bar{\chi}}\int_0^{\bar{\chi}} d\tilde{\chi} (\bar{\chi} - \tilde{\chi}) \left[ -2\Phi\tilde{\partial}_{\perp m}S_{\perp}^{m(1)} + 2\left(\frac{d\Phi}{d\tilde{\chi}} - \frac{1}{\tilde{\chi}}\Phi\right)\kappa^{(1)} \right] \left. \right\} \\
& - (24\Phi_o v_{\parallel o} + v_{\perp k o} v_{\perp o}^k) - 8v_{\parallel o} \left( \frac{1}{\bar{\chi}}T^{(1)} + 3\int_0^{\bar{\chi}} \frac{d\tilde{\chi}}{\bar{\chi}}\Phi \right) + 2(\Phi_o - v_{\parallel o}) \left[ \left( \frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \partial_{\parallel}^2 v + \left( -\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}} \right) \Phi' \right. \\
& + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^2\Phi + \frac{1}{\mathcal{H}^2}\partial_{\parallel}^3 v + \frac{1}{\mathcal{H}}\partial_{\parallel}\Phi - \frac{1}{\mathcal{H}^2}\frac{d\Phi'}{d\bar{\chi}} \left. \right] + v_{\perp o}^i \left[ -2\bar{\chi}\partial_{\perp i}\Phi + 2\frac{\bar{\chi}}{\mathcal{H}}\partial_{\perp i}(-\Phi' + \partial_{\parallel}^2 v) - 2\bar{\chi}\partial_{\perp i}\delta_g^{(1)} \right] \\
& + v_{\perp i o} \left( 4S_{\perp}^i - \frac{2}{\mathcal{H}}\partial_{\perp}^i \Delta \ln a^{(1)} + 4\bar{\chi}\partial_{\perp}^i \Phi \right)
\end{aligned} \tag{B2}$$

## 2. $Q = 1, Q^{(1)} = 0$

At first order

$$\Delta_g^{(1)} = \delta_g^{(1)} + \left( b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2 \right) \Delta \ln a^{(1)} + \Phi - \frac{1}{\mathcal{H}}\partial_{\parallel}^2 v + \frac{1}{\mathcal{H}}\Phi', \tag{B3}$$



and, at second order,

$$\begin{aligned}
\Delta_g^{(2)} = & \delta_g^{(2)} + \left(b_e - 2 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \Delta \ln a^{(2)} + \Phi^{(2)} + \frac{1}{\mathcal{H}} \Psi^{(2)'} - \frac{1}{2\mathcal{H}} \hat{h}_{\parallel}^{(2)'} - \frac{1}{\mathcal{H}} \partial_{\parallel}^2 v^{(2)} - \frac{1}{\mathcal{H}} \partial_{\parallel} \hat{v}_{\parallel}^{(2)} + 2\Phi \delta_g^{(1)} - \frac{2}{\mathcal{H}} \delta_g^{(1)} \partial_{\parallel}^2 v \\
& + \frac{2}{\mathcal{H}} \delta_g^{(1)} \Phi' + \frac{2}{\mathcal{H}} \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \Phi \Phi' + 3\Phi^2 + 2\Phi \partial_{\parallel} v + \left(3 + \frac{4}{\bar{\chi}\mathcal{H}}\right) (\partial_{\parallel} v)^2 - \frac{2}{\mathcal{H}} \left(2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \Phi \partial_{\parallel}^2 v + \frac{2}{\mathcal{H}^2} (\Phi')^2 \\
& + \frac{2}{\mathcal{H}^2} (\partial_{\parallel}^2 v)^2 + \frac{2}{\mathcal{H}^2} \partial_{\parallel} v \partial_{\parallel}^2 \Phi + \frac{4}{\mathcal{H}} \partial_{\parallel} v \partial_{\parallel} \Phi - \frac{2}{\mathcal{H}^2} \Phi \partial_{\parallel}^3 v - \frac{2}{\mathcal{H}} \Phi \partial_{\parallel} \Phi + \frac{2}{\mathcal{H}^2} \Phi \frac{d\Phi'}{d\bar{\chi}} - \frac{2}{\mathcal{H}^2} \partial_{\parallel} v \frac{d\Phi'}{d\bar{\chi}} + \frac{2}{\mathcal{H}} \left(4 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \partial_{\parallel} v \partial_{\parallel}^2 v \\
& - \frac{2}{\mathcal{H}^2} \Phi \partial_{\parallel}^2 \Phi + \frac{2}{\mathcal{H}} \left(-\frac{\mathcal{H}'}{\mathcal{H}^2}\right) \partial_{\parallel} v \Phi' - \frac{4}{\mathcal{H}^2} \partial_{\parallel}^2 v \Phi' + \frac{2}{\mathcal{H}} \partial_{\perp i} v \partial_{\perp}^i \Phi - \frac{2}{\mathcal{H}} \partial_{\perp i} v \partial_{\perp}^i \partial_{\parallel} v + \left(-1 + \frac{2}{\bar{\chi}\mathcal{H}}\right) \partial_{\perp i} v \partial_{\perp}^i v + \frac{2}{\mathcal{H}^2} \partial_{\parallel} v \partial_{\parallel}^3 v \\
& + \frac{2}{\mathcal{H}} \partial_{\parallel} v \nabla_{\perp}^2 v + \left[2\left(-6 + b_e + \frac{4}{\bar{\chi}\mathcal{H}}\right) \Phi + 2\frac{\mathcal{H}'}{\mathcal{H}^2} \partial_{\parallel} v + 2\left(b_e - 2 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \delta_g^{(1)} - \frac{2}{\mathcal{H}} \frac{d\delta_g^{(1)}}{d\bar{\chi}} + \frac{2}{\mathcal{H}} \left(-b_e + 2 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \partial_{\parallel}^2 v \right. \\
& + \frac{2}{\mathcal{H}} \left(-2 + b_e - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \Phi' - \frac{4}{\mathcal{H}} \partial_{\parallel} \Phi + 8\left(-1 + \frac{1}{\bar{\chi}\mathcal{H}}\right) \left(\frac{1}{\bar{\chi}} T^{(1)} + \kappa^{(1)}\right) \left. \right] \Delta \ln a^{(1)} + \left[-5b_e + b_e^2 + \frac{\partial b_e}{\partial \ln \bar{a}} + 10 \right. \\
& + (5 - 2b_e) \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + 3\left(\frac{\mathcal{H}'}{\mathcal{H}^2}\right)^2 + \frac{4}{\bar{\chi}^2 \mathcal{H}^2} - \frac{8}{\bar{\chi}\mathcal{H}} \left. \right] (\Delta \ln a^{(1)})^2 + 4\left[\frac{1}{\mathcal{H}} \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \Phi' + \frac{1}{\mathcal{H}} \partial_{\parallel} \Phi + \frac{1}{\mathcal{H}} \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \partial_{\parallel}^2 v \right. \\
& + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^2 \Phi + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^3 v - \frac{1}{\mathcal{H}^2} \frac{d\Phi'}{d\bar{\chi}} \left. \right] I^{(1)} + 2\left(-\partial_{\parallel} \delta_g^{(1)} + \frac{4}{\bar{\chi}} \Phi - \partial_{\parallel} \Phi + \frac{1}{\mathcal{H}} \partial_{\parallel}^3 v - \frac{1}{\mathcal{H}} \partial_{\parallel} \Phi'\right) T^{(1)} \\
& + 4\left[\frac{1}{\bar{\chi}^2} (T^{(1)})^2 + \frac{2}{\bar{\chi}} T^{(1)} \kappa^{(1)}\right] + 8\Phi \kappa^{(1)} + 4(\kappa^{(1)})^2 + 4\left[\frac{\bar{\chi}}{\mathcal{H}} (\partial_{\perp i} \Phi' - \partial_{\perp i} \partial_{\parallel}^2 v) + \bar{\chi} \partial_{\perp i} \delta_g^{(1)} + \bar{\chi} \partial_{\perp i} \Phi\right] S_{\perp}^{i(1)} \\
& + 2\left[\frac{2}{\bar{\chi}\mathcal{H}} \partial_{\perp i} v - \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} \Phi' + \frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} \partial_{\parallel}^2 v - \frac{2}{\mathcal{H}} \partial_{\perp i} \partial_{\parallel} v - \bar{\chi} \partial_{\perp i} \delta_g^{(1)} - \bar{\chi} \partial_{\perp i} \Phi\right] \partial_{\perp}^i T^{(1)} + 2(\Phi_o - v_{\parallel o}) \left[\left(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \partial_{\parallel}^2 v \right. \\
& + \left(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\right) \Phi' + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^2 \Phi + \frac{1}{\mathcal{H}^2} \partial_{\parallel}^3 v + \frac{1}{\mathcal{H}} \partial_{\parallel} \Phi - \frac{1}{\mathcal{H}^2} \frac{d\Phi'}{d\bar{\chi}} \left. \right] + v_{\perp o}^i \left[-2\bar{\chi} \partial_{\perp i} \Phi + 2\frac{\bar{\chi}}{\mathcal{H}} \partial_{\perp i} (-\Phi' + \partial_{\parallel}^2 v) - 2\bar{\chi} \partial_{\perp i} \delta_g^{(1)}\right]
\end{aligned} \tag{B4}$$

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